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Remarks on the connection between the additive separation of the Hamilton–Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators

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The commutation relations of the first-order and second-order operators associated with the first integrals in involution of a Hamiltonian separable system are examined. It is shown that these operators commute if and only if a ''pre-Robertson condition'' is satisfied. This condition involves the Ricci tensor of the configuration manifold and it is implied by the Robertson condition, which is necessary and sufficient for the separability of the Schrödinger equation. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1506181]

I. INTRODUCTION

The connection between the additive separation of the Hamilton–Jacobi equation and the multiplicative separation of the corresponding Schrödinger equation has been examined in paper 1.[1] Two different kinds of separation have been introduced for the Schrödinger equation, called ''free'' and ''reduced separation,'' respectively, related to two suitable completeness conditions for a separated solution and geometrically characterized in terms of ''Killing–Stäckel algebras'' and of ''separable Killing algebras.'' These are linear spaces of Killing tensors and Killing vectors which generate complete systems of first integrals in involution, and which characterize the separation of the Hamilton–Jacobi equation in orthogonal and in standard coordinates, respectively. The corresponding Schrödinger equation is then separable in the same coordinate system if and only if a ''Robertson condition'' is satisfied. This condition involves the Ricci tensor of the configuration manifold and it is fulfilled in the most common applications of the theory (for instance, on Einstein manifolds).

In the present paper we revisit the matter relating the separation of the Schrödinger equation to the existence of ''symmetry operators,''[2] i.e., to the existence of linear second-order operators on wave functions which commute with the Schrödinger operator. These operators are in one-to-one correspondence with the quadratic first integrals associated with the separation. We shall show that the ''quantization problem,'' i.e., the problem of defining a correspondence between classical observables and linear operators preserving the commutation relations,[3] is solvable for the involutive algebra of first integrals associated with the separation of the Hamilton–Jacobi equation provided a ''pre-Robertson'' condition is satisfied. This condition is implied by the Robertson condition, so that the quantization problem for a classical natural Hamiltonian system is solvable if the corresponding Schrödinger equation is separable. The main theorems and remarks are stated in Secs. III and VI in the case of the orthogonal and general separation, respectively. The proofs are given in Secs. V and VIII, after general considerations about Killing tensors in orthogonal and standard form illustrated in Secs. IV and VII.

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II. GENERAL COMMUTATION RELATIONS FOR SECOND-ORDER DIFFERENTIAL LINEAR OPERATORS

By the "quantization problem"[3] we mean the problem of defining a correspondence $F \mapsto \hat{F}$ between classical and quantum observables, i.e., between smooth real functions on the cotangent bundle T^*Q (the "phase space") of the configuration manifold of a mechanical system and self-adjoint linear operators on a suitable "state-space" of complex-valued functions (or distributions) on Q. This correspondence is required to be $\mathbb{R}$-linear and preserving the Lie-algebra structure of classical and quantum observables:

$$(F+G)\hat{} = \hat{F}+\hat{G}, \quad (cF)\hat{} = c\hat{F} \quad (c \in \mathbb{R}), \quad \{F,G\}\hat{} = \gamma[\hat{F},\hat{G}].$$

Here, $\{F,G\}$ denotes the canonical Poisson bracket of functions, γ is a universal constant, and

$$[\hat{F},\hat{G}] = \hat{F}\hat{G} - \hat{G}\hat{F}$$

is the commutator of linear operators.

The quantization problem is not solvable on the whole set of observables of a phase space.[4–6] (see Ref. 3 for details, comments, and references). In accordance with Schwinger (Ref. 7, Sec. 2.4) we can say not only that "it is a convenient fiction to assert that every Hermitian operator symbolizes a physical quantity [...]" but also that it is a "convenient fiction" to assert that with every classical observable we can associate an Hermitian operator (i.e., a quantum observable). However, as we shall see, the quantization problem is solvable for the classical observables involved in the separation of variables of a natural Hamiltonian system, which are polynomials of second degree in the momenta (p_i).

We consider as a starting point the following assumptions: (i) The universal constant γ is a positive-imaginary number: $\gamma = i/h$, $\hbar \in \mathbb{R}_+$. (ii) The operator $\hat{f}$ corresponding to a function f on Q, interpreted as a function on T^*Q constant on the fibers, is defined by

$$\hat{f}\psi = f \cdot \psi.$$

As usual, the operator $\hat{f}$ will be simply denoted by f. (iii) The operator $\hat{P}_{\mathbf{X}}$ corresponding to a first-degree homogeneous polynomial

$$P_{\mathbf{X}} = X^i p_i$$

associated with a vector field $\mathbf{X}$ on Q, is defined by

$$\hat{P}_{\mathbf{X}}\psi = \frac{1}{\gamma}\langle \mathbf{X}, \mathrm{d}\psi \rangle = -i\hbar \langle \mathbf{X}, \mathrm{d}\psi \rangle.$$

(iv) The operator $\hat{P}_{\mathbf{K}}$ corresponding to a second-degree homogeneous polynomial

$$P_{\mathbf{K}} = K^{ij} p_i p_j$$

associated with a symmetric contravariant two-tensor $\mathbf{K}$ on Q, is defined by

$$\hat{P}_{\mathbf{K}}\psi = -\hbar^2 \Delta_{\mathbf{K}}\psi = -\hbar^2 \nabla_i (K^{ij} \nabla_j \psi),$$

where $\Delta_{\mathbf{K}}$ is the *pseudo-Laplacian* operator defined by

$$\Delta_{\mathbf{K}}\psi = \nabla_i (K^{ij} \nabla_j \psi) \tag{2.1}$$

(by ∇_i we denote the covariant derivative with respect to the Levi-Civita connection). For $\mathbf{K} = \mathbf{G}$ (the contravariant metric tensor) we find the Laplace–Beltrami operator $\Delta_{\mathbf{G}} = \Delta$,

$$\Delta\psi = g^{ij}\nabla_i\nabla_j\psi.$$

We shall use the *co-differential or divergence operator* δ on contravariant skew-symmetric tensors $\mathbf{A}$, defined by

$$(\delta\mathbf{A})^{j\cdots k} = \nabla_i A^{ij\cdots k}.$$

For a function (zero-tensor), $\delta f = 0$. It follows that $\delta^2 = 0$.

According to the notation used in Ref. 1, Remark 7.1, we shall identify a (contravariant) two-tensor $\mathbf{K} = (K^{ij})$ with the corresponding linear endomorphisms on vectors and one-forms, so that we shall denote by $\mathbf{K}\nabla\psi$ the vector field image of the gradient $\nabla\psi$ (whose components are $K^{ij}\partial_j\psi$) and by $\mathbf{K}\,d\psi$ the one-form image of the differential $d\psi$ (whose components are $g_{ih}K^{hj}\partial_j\psi$). With this notation, the coordinate independent definition of the pseudo-Laplacian (2.1) is

$$\Delta_{\mathbf{K}}\psi = \delta(\mathbf{K}\nabla\psi).$$

We shall deal with quadratic classical observables of the kind

$$H_{\mathbf{K}} = \tfrac{1}{2}P_{\mathbf{K}} + V_{\mathbf{K}}, \quad V_{\mathbf{K}}: Q \to \mathbb{R},$$

and with the corresponding second-order operators

$$\hat{H}_{\mathbf{K}} = \frac{1}{2}\hat{P}_{\mathbf{K}} + V_{\mathbf{K}} = -\frac{\hbar^2}{2}\Delta_{\mathbf{K}} + V_{\mathbf{K}}. \tag{2.2}$$

For $\mathbf{K} = \mathbf{G}$ we find the *Hamiltonian* and the *Schrödinger operator*,

$$H = H_G = \frac{1}{2}P_{\mathbf{G}} + V, \quad \hat{H} = \frac{1}{2}\hat{P}_{\mathbf{G}} + V = -\frac{\hbar^2}{2}\Delta + V.$$

A classical observable F in involution with H, $\{F,H\} = 0$, is a *first integral* (or *constant of motion*) of the Hamiltonian system generated by H. A linear operator $\hat{F}$ commuting with $\hat{H}$,

$$[\hat{F},\hat{H}] = 0,$$

is called a *symmetry operator* of the Schrödinger equation. The following commutation rules hold for these classical observables,

$$\begin{aligned} \{H_{\mathbf{K}_1}, H_{\mathbf{K}_2}\} &= \{P_{\mathbf{K}_1}, P_{\mathbf{K}_2}\} + P_{\mathbf{K}_1\nabla V_{\mathbf{K}_2}} - P_{\mathbf{K}_2\nabla V_{\mathbf{K}_1}}, \\ \{H_{\mathbf{K}}, H\} &= \{P_{\mathbf{K}}, P_{\mathbf{G}}\} + P_{\mathbf{K}\nabla V} - P_{\nabla V_{\mathbf{K}}}. \end{aligned} \tag{2.3}$$

We recall that a *Killing tensor* is a symmetric tensor (of any order) satisfying one of the following two equivalent conditions:

$$\{P_{\mathbf{K}}, P_G\} = 0 \Leftrightarrow \nabla^{(i}K^{j\cdots k)} = 0, \tag{2.4}$$

where $P_{\mathbf{K}} = K^{ij\cdots k}p_i p_j \ldots p_k$ and the brackets $(\cdots)$ denote the symmetrization of the indices. The first equation (2.4) means that $P_{\mathbf{K}}$ is a first integral of the godesic flow.

In the second equation (2.3) the term $\{P_{\mathbf{K}}, P_{\mathbf{G}}\}$ is a third-degree homogeneous polynomial in the momenta (p_i), while the remaining term is of first degree. This shows that

Theorem 2.1: *The quadratic function $H_{\mathbf{K}}$ is a first integral of the Hamiltonian flow generated by H if and only if $\mathbf{K}$ is a Killing tensor and $\mathbf{K}\,dV = dV_{\mathbf{K}}$ i.e., the following conditions are equivalent*

$$\{H_{\mathbf{K}},H\}=0 \Leftrightarrow \begin{array}{c}\{P_{\mathbf{K}},P_{\mathbf{G}}\}=0 \quad (\mathbf{K}\text{ Killing tensor}),\\ \nabla V_{\mathbf{K}}=\mathbf{K}\nabla V.\end{array} \tag{2.5}$$

For the related operators we have

Theorem 2.2: *The following conditions are equivalent*

$$[\hat{H}_{\mathbf{K}},\hat{H}]=0 \Leftrightarrow \begin{array}{c}\{P_{\mathbf{K}},G_{\mathbf{G}}\}=0 \quad (\mathbf{K}\text{ Killing tensor}),\\ \mathbf{K}\nabla V-\nabla V_{\mathbf{K}}+\dfrac{\hbar^2}{6}\,\delta\mathbf{C}=0,\end{array} \Leftrightarrow \{H_{\mathbf{K}},H\}=-\frac{\hbar^2}{6}P_{\delta\mathbf{C}}, \tag{2.6}$$

where

$$\mathbf{C}=\mathbf{KR}-\mathbf{RK},\quad C^{ij}=K^{ih}R^j_h-R^{ih}K^j_h=K^{ih}g_{hk}R^{kj}-R^{ih}g_{hk}K^{kj},\quad C^{ij}=-C^{ji},$$

and $\mathbf{R}$ *is the Ricci tensor:*

Proof: In accordance with the above-given definitions we have

$$\begin{aligned}\hat{H}\hat{H}_{\mathbf{K}}\psi&=-\frac{\hbar^2}{2}\Delta\left(-\frac{\hbar^2}{2}\Delta_{\mathbf{K}}\psi+V_{\mathbf{K}}\psi\right)+V\left(-\frac{\hbar^2}{2}\Delta_{\mathbf{K}}\psi+V_{\mathbf{K}}\psi\right)\\&=\frac{\hbar^4}{4}\Delta\Delta_{\mathbf{K}}\psi-\frac{\hbar^2}{2}(\Delta(V_{\mathbf{K}}\psi)+V\Delta_{\mathbf{K}}\psi)+VV_{\mathbf{K}}\psi,\end{aligned}$$

$$\hat{H}_{\mathbf{K}}\hat{H}\psi=\frac{\hbar^4}{4}\Delta_{\mathbf{K}}\Delta\psi-\frac{\hbar^2}{2}(\Delta_{\mathbf{K}}(V\psi)+V_{\mathbf{K}}\Delta\psi)+V_{\mathbf{K}}V\psi,$$

$$\Delta_{\mathbf{K}}(V\psi)=\psi\Delta_{\mathbf{K}}V+2\mathbf{K}(\mathrm{d}V,\mathrm{d}\psi)+V\Delta_{\mathbf{K}}\psi,$$

$$\Delta(V_{\mathbf{K}}\psi)=\psi\Delta V_{\mathbf{K}}+2\mathbf{G}(\mathrm{d}V_{\mathbf{K}},\mathrm{d}\psi)+V_{\mathbf{K}}\Delta\psi.$$

Hence,

$$\begin{aligned}[\hat{H},\hat{H}_{\mathbf{K}}]\psi&=\frac{\hbar^4}{4}[\Delta,\Delta_{\mathbf{K}}]\psi+\frac{\hbar^2}{2}(\psi\Delta_{\mathbf{K}}V+2\mathbf{K}(\mathrm{d}V,\mathrm{d}\psi)-\psi\Delta V_{\mathbf{K}}-2\mathbf{G}(\mathrm{d}V_{\mathbf{K}},\mathrm{d}\psi))\\&=\frac{\hbar^4}{4}[\Delta,\Delta_{\mathbf{K}}]\psi+\hbar^2(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}})\cdot\nabla\psi+\frac{\hbar^2}{2}(\Delta_{\mathbf{K}}V-\Delta V_{\mathbf{K}})\psi.\end{aligned} \tag{2.7}$$

Now we use a formula due to Carter[8] which gives an explicit expression of the commutator of a pseudo-Laplacian with the ordinary Laplacian,

$$\begin{aligned}[\Delta,\Delta_{\mathbf{K}}]\psi&=2\nabla^hK^{ij}\nabla_{(i}\nabla_j\nabla_{h)}\psi+3\nabla_h\nabla^{(h}K^{ij)}\nabla_{(i}\nabla_{j)}\psi\\&\quad+\nabla_j\left(\tfrac{1}{2}g_{hk}(\nabla^j\nabla^{(i}K^{hk)}-\nabla^i\nabla^{(j}K^{hk)})+\tfrac{4}{3}K^{[j}_hR^{i]h}\right)\nabla_i\psi,\end{aligned} \tag{2.8}$$

where the brackets $[\cdots]$ denote the skew-symmetrization of the indices. Gathering together and equating to zero the terms of third, first-, and zero-order derivatives of ψ on the right-hand side of (2.7) we get the following equations, respectively,

$$\nabla^{(h}K^{ij)}=0 \quad (\mathbf{K}\text{ is a Killing tensor}),$$

$$\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}+\frac{\hbar^2}{6}\,\delta(\mathbf{KR}-\mathbf{RK})=0, \tag{2.9}$$

$$\Delta_{\mathbf{K}}V-\Delta V_{\mathbf{K}}=0.$$

The second-order terms in (2.7) disappear because of the first equation (2.9). The last equation (2.9) can be written $\delta(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}})=0$, so that it becomes a consequence of the second equation, since $\mathbf{C}=\mathbf{KR}-\mathbf{RK}$ is skew-symmetric and $\delta^2=0$. This proves the first equivalence (2.6). The second equivalence follows from the last equation (2.3), since $\{P_{\mathbf{K}},P_{\mathbf{G}}\}$ is a homogeneous polynomial of third degree in (p_i), while $P_{\mathbf{K}\nabla V}-P_{\nabla V_{\mathbf{K}}}$ and $P_{\delta\mathbf{C}}$ are of first degree. ■

The following three propositions are a consequence of Theorem 2.2 and of the *Carter formula* (2.8).

Proposition 2.3: If $\mathbf{K}=(K^{ij})$ *is a symmetric tensor, then* $[\hat{P}_{\mathbf{K}},\hat{P}_{\mathbf{G}}]=0$ *if and only if* $\mathbf{K}$ *is a Killing tensor and*

$$\delta\mathbf{C}=\delta(\mathbf{KR}-\mathbf{RK})=0. \tag{2.10}$$

Proof: This is a special case of the first equivalence (2.6), for $V=0$ and $V_{\mathbf{K}}=0$. ■

We call (2.10) the *Carter condition*. Note that $[\hat{P}_{\mathbf{K}},\hat{P}_{\mathbf{G}}]=0$ is equivalent to $[\Delta_{\mathbf{K}},\Delta]=0$.

Proposition 2.4: If $\mathbf{K}$ *is a Killing tensor, then*

$$\{H,H_{\mathbf{K}}\}=0 \Rightarrow [\hat{H},\hat{H}_{\mathbf{K}}]\psi=\frac{\hbar^4}{6}\,\delta\mathbf{C}\cdot\nabla\psi. \tag{2.11}$$

Proof: For a Killing tensor the Carter formula (2.8) reduces to

$$[\Delta,\Delta_{\mathbf{K}}]\psi=\tfrac{2}{3}\delta\mathbf{C}\cdot\nabla\psi$$

so that (2.7) becomes

$$[\hat{H},\hat{H}_{\mathbf{K}}]\psi=\frac{\hbar^4}{6}\,\delta C\cdot\nabla\psi+\hbar^2(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}})+\frac{\hbar^2}{2}\,\delta(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}).$$

Because of the equivalence (2.5), we get the second equation (2.11). ■

Proposition 2.5: Let $H_{\mathbf{K}}=\frac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}}$ *be a quadratic first integral i.e.,* $\{H_{\mathbf{K}},H\}=0$. *Then,* $[\hat{H}_{\mathbf{K}},\hat{H}]=0$ *if and only if the Carter condition (2.10) is satisfied.*

Proof: If (2.10) holds, then $[\hat{H}_{\mathbf{K}},\hat{H}]=0$ because of the implication (2.11). Conversely, the simultaneous conditions $[\hat{H}_{\mathbf{K}},\hat{H}]=0$ and $\{H_{\mathbf{K}},H\}=0$ imply $\delta\mathbf{C}=0$ because of the equivalence (2.6) ■

As a corollary of Theorem 2.2 we have

Theorem 2.6: *If* $\mathbf{R}=\kappa\mathbf{G}$, *then*

$$\{H_{\mathbf{K}},H\}=0 \Leftrightarrow [\hat{H}_{\mathbf{K}},\hat{H}]=0.$$

This shows that on Einstein manifolds (in particular, on manifolds with constant curvature, on flat manifolds, on Ricci-flat manifolds, etc.) a quadratic function $H_{\mathbf{K}}=\frac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}}$ is a first integral if and only if the corresponding operator $\hat{H}_{\mathbf{K}}$, defined according to (2.1) and (2.2), is a symmetry of the Schrödinger equation.

For a first-order operator $\hat{P}_{\mathbf{X}}$ we have a similar equivalence, but without any condition (like the Carter condition) involving the Ricci tensor:

Theorem 2.7: *The operator* $\hat{P}_{\mathbf{X}}$ *commutes with the Laplacian if and only if* $\mathbf{X}$ *is a Killing vector*,

$$[\hat{P}_{\mathbf{X}},\Delta]=0 \Leftrightarrow \{P_{\mathbf{X}},P_{\mathbf{G}}\}=0.$$

Proof: This follows from three basic facts. (i) A vector field is a Killing vector if and only if its covariant components satisfy equation

$$\nabla_i X_j + \nabla_j X_i = 0. \tag{2.12}$$

This is in accordance with (2.4). (ii) If $\mathbf{X} = (X^i)$ is a Killing vector, then[9]

$$\Delta X^i + R^i_j X^j = 0, \tag{2.13}$$

where R_{ij} is the Ricci tensor. This follows from the general commutation rule

$$\nabla_l \nabla_k X_i - \nabla_k \nabla_l X_i = X_m R^m_{\cdot ikl}, \tag{2.14}$$

where $R^m_{\cdot ikl}$ are the components of the Riemann tensor. Indeed, by setting $g^{il} R^m_{\cdot ikl} = R^m_k$, we get

$$g^{il}(\nabla_l \nabla_k X_i - \nabla_k \nabla_l X_i) = R^m_k X_m .$$

For a Killing vector, $\nabla_k X_i$ is skew-symmetric due to (2.12) thus,

$$-g^{il} \nabla_l \nabla_i X_k = R^m_k X_m ,$$

and this equation is equivalent to (2.13). (iii) For any vector field $\mathbf{X}$, the general commutation relation

$$[\mathbf{X}, \Delta]\psi = -(\Delta X^l + X^i R^l_i)\nabla_l \psi - 2\nabla^h X^i \nabla_h \nabla_i \psi, \tag{2.15}$$

holds, where $\mathbf{X}(\psi) = X^i \nabla_i \psi$. Indeed,

$$\begin{aligned}[\mathbf{X}, \Delta]\psi &= [X^i \nabla_i , g^{hk} \nabla_h \nabla_k]\psi \\ &= X^i g^{hk} \nabla_i \nabla_h \nabla_k \psi - g^{hk} \nabla_h \nabla_k (X^i \nabla_i \psi) \\ &= g^{hk}(X^i \nabla_i \nabla_h \nabla_k \psi - X^i \nabla_h \nabla_k \nabla_i \psi - \nabla_h \nabla_k X^i \nabla_i \psi - \nabla_k X^i \nabla_h \nabla_i \psi - \nabla_h X^i \nabla_k \nabla_i \psi)\end{aligned} \tag{2.16}$$

However, because of (2.14),

$$\nabla_i \nabla_h \nabla_k \psi = \nabla_h \nabla_i \nabla_k \psi = \nabla_j \psi R^j_{\cdot khi} = \nabla_h \nabla_k \nabla_i \psi + \nabla_j \psi R^j_{\cdot khi} ,$$

since $\nabla_i \nabla_k \psi$ is symmetric. Thus, the last expression (2.16) becomes

$$[\mathbf{X}, \Delta]\psi = g^{hk} X^i R^j_{\cdot khi} \nabla_j \psi - \Delta X^i \nabla_i \psi - 2\nabla^h X^i \nabla_h \nabla_i \psi$$

and (2.15) is proved. Assume that $\mathbf{X}$ is a Killing vector. Then the first term on the right-hand side of (2.15) vanishes because of (2.13), as well as the second term, since $\nabla^h X^i$ is skew-symmetric because of (2.12). Conversely, assume that (2.15) is satisfied for all functions ψ. Then the coefficients of the first and second derivatives of ψ must vanish separately. The coefficients of the second derivatives yield equation $\nabla^h X^i \partial_h \partial_i \psi = 0$, which shows that $\nabla^h X^i$ is skew-symmetric. Thus, $\mathbf{X}$ is a Killing vector according to (2.12), and the first-order terms vanish due to (2.13). ■

From Theorem 2.7 it follows that

Theorem 2.8: *The operator $\hat{P}_{\mathbf{X}}$ commutes with the Schrödinger operator $\hat{H}$ if and only if* $\mathbf{X}$ *is a Killing vector and* $\langle \mathbf{X}, dV \rangle = 0$ *i.e.*,

$$[\hat{P}_{\mathbf{X}}, \hat{H}] = 0 \Leftrightarrow \{P_{\mathbf{X}}, H\} = 0. \tag{2.17}$$

Proof: Since

$$[\hat{P}_{\mathbf{X}}, \tfrac{1}{2}\hat{P}_{\mathbf{G}} + V]\psi = \tfrac{1}{2}[\hat{P}_{\mathbf{X}}, \hat{P}_{\mathbf{G}}]\psi + \hat{P}_{\mathbf{X}}(V\psi) - V\hat{P}_{\mathbf{X}}\psi = \tfrac{1}{2}[\hat{P}_{\mathbf{X}}, \hat{P}_{\mathbf{G}}]\psi + \hat{P}_{\mathbf{X}}(V)\psi,$$

the first commutation relation (2.17) is equivalent to

$$[\hat{P}_{\mathbf{X}},\Delta]=0,\quad \hat{P}_{\mathbf{X}}(V)=0.$$

Moreover,

$$\{P_{\mathbf{X}},\tfrac{1}{2}P_{\mathbf{G}}+V\}=\tfrac{1}{2}\{P_{\mathbf{X}},P_{\mathbf{G}}\}+\{P_{\mathbf{X}},V\}=\tfrac{1}{2}\{P_{\mathbf{X}},P_{\mathbf{G}}\}+\langle\mathbf{X},\mathrm{d}V\rangle,$$

and the second commutation relation (2.17) is equivalent to

$$\{P_{\mathbf{X}},P_{\mathbf{G}}\}=0,\quad \langle\mathbf{X},\mathrm{d}V\rangle=0.$$

Thus, the two commutation relations are equivalent due to Theorem 2.7. ■

III. SYMMETRY OPERATORS ASSOCIATED WITH THE ORTHOGONAL SEPARATION OF THE HAMILTON–JACOBI EQUATION

A *Killing–Stäckel algebra* is an n-dimensional linear space $\mathcal{K}$ of Killing two-tensors with common normal eigenvectors.[1] It can be proved that $\mathbf{G}\in\mathcal{K}$ and that all functions $P_{\mathbf{K}}$, $\mathbf{K}\in\mathcal{K}$, are in involution. The Hamilton–Jacobi equation associated with a natural Hamiltonian

$$H=\tfrac{1}{2}P_{\mathbf{G}}+V=\tfrac{1}{2}g^{ij}(\underline{q})p_ip_j+V(\underline{q})$$

is separable (i.e., integrable by separation of variables) in orthogonal coordinates if and only if there exists a Killing–Stäckel algebra such that equation $\mathrm{d}(\mathbf{K}\,\mathrm{d}V)=0$ is satisfied for all $\mathbf{K}\in\mathcal{K}$, or for a single Killing tensor $\mathbf{K}$ with simple eigenvalues (such a tensor is called a *characteristic tensor* of $\mathcal{K}$). Then: (i) The separation occurs in any coordinate system (q^i) such that $\mathrm{d}q^i$ are (common) eigenforms of the elements of $\mathcal{K}$. In these coordinates all elements of $\mathcal{K}$ are diagonalized,

$$\mathbf{K}=K^{ii}\partial_i\otimes\partial_i=\lambda^i g^{ii}\partial_i\otimes\partial_i, \tag{3.1}$$

λ^i being the eigenvalues of $\mathbf{K}$ (for $\mathbf{K}=\mathbf{G}$ we have $\lambda^i=1$). (ii) There are local functions $V_{\mathbf{K}}$ on Q such that $\mathrm{d}V_{\mathbf{K}}=\mathbf{K}\,\mathrm{d}V$ or

$$\nabla V_{\mathbf{K}}=\mathbf{K}\nabla V \tag{3.2}$$

for all $\mathbf{K}\in\mathcal{K}$. It follows that the functions

$$H_{\mathbf{K}}=\tfrac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}},\quad \mathbf{K}\in\mathcal{K},$$

are first integrals in involution,

$$\{H_{\mathbf{K}_1},H_{\mathbf{K}_2}\}=0,\quad \forall\mathbf{K}_1,\mathbf{K}_2\in\mathcal{K}.$$

We denote by

$$\mathcal{H}=(\mathcal{K},V)$$

the n-dimensional space of these first integrals determined by a Killing-Stäckel algebra $\mathcal{K}$ and by a potential V satisfying (3.2).

In general, the linear operators $\hat{H}_{\mathbf{K}}$ (2.2) corresponding to these quadratic first integrals do not commute, as shown by the following

Theorem 3.1: *Let $\mathcal{H}=(\mathcal{K},V)$ be the space of first integrals in involution associated with the orthogonal separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent*

$$\begin{aligned}
&\text{(a)}\ \ [\hat{H}_{\mathbf{K}},\hat{H}]=0, \quad \forall \mathbf{K}\in\mathcal{K},\\
&\text{(b)}\ \ \delta(\mathbf{KR}-\mathbf{RK})=0, \quad \forall \mathbf{K}\in\mathcal{K},\\
&\text{(c)}\ \ \partial_i R_{ij}-\Gamma_i R_{ij}=0, \quad i\neq j,\ \ i \ \text{ n.s.,}\\
&\text{(d)}\ \ [\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2,\in\mathcal{K},\\
&\text{(e)}\ \ \delta(\mathbf{K}_1\mathbf{R}\mathbf{K}_2-\mathbf{K}_2\mathbf{R}\mathbf{K}_1)=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2,\in\mathcal{K},
\end{aligned} \tag{3.3}$$

where $\mathbf{R}$ *is the Ricci tensor*, R_{ij} *are its components in any orthogonal separable coordinate system and* Γ_i *are the contracted Christoffel symbols*, $\Gamma_i=g^{hj}\Gamma_{hj,i}$.

The proof will be given in Sec. V. In Sec. VI, a theorem analogous to Theorem 3.1 will be stated for the general nonorthogonal separation of the Hamilton–Jacobi equation, where the Killing tensors involved are in "standard form." For the proofs of these theorems we need preliminary general results about Killing tensors and second-order operators. Indeed, since we do not know how to extend the Carter formula to two arbitrary symmetric tensors, we are able to study the commutator $[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]$ only for Killing tensors in orthogonal form (Secs. IV and V) or in standard form (Secs. VII and VIII).

Remark 3.2: Note that conditions (b), (c), and (e) in (3.3) do not involve the potentials $V_{\mathbf{K}}$.

Remark 3.3: Since $\mathbf{G}\in\mathcal{K}$, equation (3.3a) is an obvious consequence of (3.3d), while (3.3b) is a consequence of (3.3e). Moreover, the equivalence of (3.3a) and (3.3b) follows from Theorems 2.1 and 2.2. We call condition (3.3b),

$$\delta(\mathbf{KR}-\mathbf{RK})=0, \quad \forall \mathbf{K}\in\mathcal{K}$$

the *pre-Robertson condition*. It means that the Carter condition (2.10) is satisfied by all elements of the Killing–Stäckel algebra. Theorem 3.1 shows that Eq. (3.3c),

$$\partial_i R_{ij}-\Gamma_i R_{ij}=0 \quad (i\neq j,\ \ i \ \text{ n.s.})$$

is the coordinate expression of the pre-Robertson condition. The pre-Robertson condition (3.3b) is an obvious consequence of the *Robertson condition*[1]

$$\mathbf{KR}-\mathbf{RK}=0, \quad \forall \mathbf{K}\in\mathcal{K},$$

whose coordinate expression is

$$R_{ij}=0, \quad i\neq j. \tag{3.4}$$

Note that both conditions are fulfilled when $\mathbf{R}=\kappa\mathbf{G}$. We know[1] that in separable orthogonal coordinates

$$\partial_i\Gamma_j=\tfrac{2}{3}R_{ij}, \quad i\neq j, \tag{3.5}$$

and that the Schrödinger equation is freely separable if and only if the Hamilton–Jacobi equation is orthogonally separable and the Robertson condition holds. Hence,

Theorem 3.4: *If the Schrödinger equation associated with an orthogonal separable Hamiltonian system is freely separable, then all the operators* $\hat{H}_{\mathbf{K}}$ *corresponding to the quadratic first integrals in involution* $H_{\mathbf{K}}\in\mathcal{H}$ *commute.*

In particular, they commute with the Schrödinger operator $\hat{H}=\hat{H}_{\mathbf{G}}$. From Theorem 3.1 we derive an extension of Theorem 2.6,

Theorem 3.5: *On Einstein manifolds all operators* $\hat{H}_{\mathbf{K}}$, $\mathbf{K}\in\mathcal{K}$, *associated with the quadratic first integrals of an orthogonal separable system commute.*

Remark 3.6: In orthogonal separable coordinates the components of the Killing tensors and the potential functions assume the *Stäckel form*

$$g^{ii}=\varphi^i_{(n)}, \quad V=\phi_i(q^i)\varphi^i_{(n)}, \quad K^{ii}_j=\varphi^i_{(j)}, \quad V_{\mathbf{K}_j}=\phi_i(q^i)\varphi^i_{(j)},$$

where $(\mathbf{K}_j)$ is a local basis of $\mathcal{K}$, with $\mathbf{K}_n=\mathbf{G}$. Thus, in terms of Stäckel matrices, a local basis of $\mathcal{H}$ is given by

$$H_j=\tfrac{1}{2}\varphi^i_{(j)}(p_i^2+2\phi_i).$$

As it will be shown (Remark 5.2), the corresponding operators assume the form

$$\hat{H}_j\psi=-\frac{\hbar^2}{2}\varphi^i_{(j)}\left(\partial_i^2\psi-\Gamma_i\partial_i\psi-\frac{2}{\hbar^2}\phi_i\psi\right). \tag{3.6}$$

The Robertson condition is equivalent to $\partial_j\Gamma_i=0$ for $i\neq j$. This means that the contracted Christoffel symbols Γ_i are functions of the corresponding coordinate q^i only.

IV. KILLING TENSORS DIAGONALIZED IN ORTHOGONAL COORDINATES

In the next section we shall analyze the commutation relations of the second-order operators assuming that all the tensors **K** involved, including the metric tensor **G**, are simultaneously diagonalized in orthogonal coordinates (q^i), so that they assume the *orthogonal form* (3.1). This is equivalent to assume that all these tensors have common normal eigenvectors (or closed eigenforms). For this purpose we need some preliminary theorems about Killing tensors diagonalized in orthogonal coordinates. For such a Killing tensor the following equations hold:

$$\begin{aligned} \partial_i\lambda^j&=(\lambda^i-\lambda^j)\partial_i\ln g^{jj}\\ \partial_i\lambda^i&=0\\ \partial_i(\lambda^j g^{jj})&=\lambda^i\partial_i g^{jj}\\ \partial_i^2(\lambda^j g^{jj})&=\lambda^i\partial_i^2 g^{jj}. \end{aligned} \qquad (i,j \ \text{ n.s.}) \tag{4.1}$$

We call *Eisenhart–Killing equations* the first equations (4.1).[10] They characterize a Killing tensor and imply the remaining equations.

In orthogonal coordinates, the nonvanishing Christoffel symbols are

$$\begin{aligned} \Gamma^j_{ij}=\Gamma^j_{ji}&=-\tfrac{1}{2}\partial_i\ln g^{jj}, \quad i \ \text{ n.s.},\\ \Gamma^i_{jj}&=-\tfrac{1}{2}g^{ii}\partial_i g_{jj}, \quad i\neq j. \end{aligned} \tag{4.2}$$

It follows that

$$\Gamma_i=\frac{1}{2}\partial_i\sum_k \ln g^{kk}-\partial_i\ln g^{ii}, \tag{4.3}$$

and

$$\sum_i \Gamma^i_{ih}=-\Gamma_h-\partial_h\ln g^{hh}. \tag{4.4}$$

Proposition 4.1: If (q^i) are orthogonal coordinates in which a Killing tensor **K** *is diagonalized, then*

$$(\lambda^i-\lambda^j)(\partial_i\Gamma_j-\partial_j\Gamma_i)=0 \quad (i,j \ \text{ n.s.}). \tag{4.5}$$

Proof: For $\lambda^i = \lambda^j$ Eq. (4.5) is obviously satisfied. Assume $\lambda^i \neq \lambda^j$. Because of (4.3) and (4.1),

$$\begin{aligned}\partial_i \Gamma_j - \partial_j \Gamma_i &= -\partial_i \partial_j \ln g^{jj} + \partial_j \partial_i \ln g^{ii} \\ &= \partial_i \partial_j \ln g^{ii} - \partial_j \partial_i \ln g^{jj} = \partial_i \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} - \partial_j \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \\ &= \frac{1}{(\lambda^i - \lambda^j)^2} [\partial_i \partial_j \lambda^i (\lambda^j - \lambda^i) - \partial_j \lambda^i \partial_i (\lambda^j - \lambda^i) - \partial_j \partial_i \lambda^j (\lambda^i - \lambda^j) + \partial_i \lambda^j \partial_j (\lambda^i - \lambda^j)] = 0,\end{aligned}$$

since $\partial_i \lambda^i = 0$. ■

Proposition 4.2: Let $\mathbf{K}_I$, $I = 1, 2$, *be two Killing tensors simultaneously diagonalized in orthogonal coordinates. Then,*

$$(\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j)(\partial_i \Gamma_j - \partial_j \Gamma_i) = 0 \quad (i \neq j \;\; \text{n.s.}). \tag{4.6}$$

Proof: Because of (4.5),

$$(\lambda_1^i - \lambda_1^j)(\partial_i \Gamma_j - \partial_j \Gamma_i) = 0, \quad (\lambda_2^i - \lambda_2^j)(\partial_i \Gamma_j - \partial_j \Gamma_i) = 0. \tag{4.7}$$

Assume $\lambda_I^j \neq 0$. If we multiply the first equation (4.7) by λ_2^j, the second one by λ_1^j and subtract the two resulting equations, then we get (4.6). If $\lambda_2^j = 0$, then the second equation (4.7) becomes $\lambda_2^i(\partial_i \Gamma_j - \partial_j \Gamma_i) = 0$ and (4.6) is satisfied. Similarly for $\lambda_1^j = 0$. ■

Proposition 4.3: Let $\mathbf{K}_I = (K_I^{ij})$, $I = 1, 2$, *be two Killing tensors simultaneously diagonalized in orthogonal coordinates. Let us define*

$$\mathbf{C} = \mathbf{K}_1 \mathbf{D} \mathbf{K}_2 - \mathbf{K}_2 \mathbf{D} \mathbf{K}_1, \quad C^{ij} = K_1^{ih} D_{hk} K_2^{kj} - K_2^{ih} D_{hk} K_1^{kj}, \tag{4.8}$$

where $\mathbf{D} = (D_{ij})$ *is a geometrical object. Then,*

$$C^{ij} = g^{ii} g^{jj} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j) D_{ij}, \quad C_j^i = C^i_{\cdot j} = g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j) D_{ij} \tag{4.9}$$

and

$$\nabla_i C_j^i = \sum_i g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j) \left(\partial_i D_{ij} - \Gamma_i D_{ij} + \frac{1}{2} \partial_i \ln g^{jj} (D_{ji} - D_{ij}) \right). \tag{4.10}$$

Proof: In orthogonal coordinates

$$C^{ij} = K_1^{ii} D_{ij} K_2^{jj} - K_2^{ii} D_{ij} K_1^{jj}$$

and (4.9) follow from $K_I^{ii} = \lambda_I^i g^{ii}$. Moreover, by definition of covariant derivative,

$$\nabla_i C_j^i = \partial_i C_j^i + \Gamma_{ih}^i C_j^h - \Gamma_{ij}^h C_h^i. \tag{4.11}$$

We compute these three terms separately by using Eqs. (4.1). For the first term,

$$\begin{aligned}\partial_i C_j^i &= \partial_i g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j) D_{ij} + g^{ii} D_{ij} (\lambda_1^i \partial_i \lambda_2^j - \lambda_2^i \partial_i \lambda_1^j) + g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j) \partial_i D_{ij} \\ &= g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j)(D_{ij} \partial_i \ln g^{ii} + \partial_i D_{ij}) + g^{ii} D_{ij} (\lambda_1^i (\lambda_2^i - \lambda_2^j) - \lambda_2^i (\lambda_1^i - \lambda_1^j)) \partial_i \ln g^{jj} \\ &= g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j)[(\partial_i \ln g^{ii} - \partial_i \ln g^{jj}) D_{ij} + \partial_i D_{ij}].\end{aligned} \tag{4.12}$$

To compute the second term we use formula (4.4),

$$\Gamma_{ih}^i C_j^h = -g^{ii} (\lambda_1^i \lambda_2^j - \lambda_2^i \lambda_1^j)(\Gamma_i + \partial_i \ln g^{ii}) D_{ij}. \tag{4.13}$$

To compute the third term we use formulas (4.2),

$$\sum_{i,h} \Gamma^h_{ij} C^i_h = \sum_{h\neq j} \Gamma^h_{jj} C^j_h + \sum_{i\neq j} \sum_h \Gamma^h_{ij} C^i_h = \cdots .$$

Since $C^i_i = 0$ (i n.s.),

$$\begin{aligned} \cdots &= \sum_h \Gamma^h_{jj} C^j_h + \sum_{i\neq j} \Gamma^i_{ij} C^i_i + \sum_{i\neq j} \sum_{h\neq i} \Gamma^h_{ij} C^i_h \\ &= \sum_i \Gamma^i_{jj} C^j_i + \sum_{i\neq j} \Gamma^j_{ij} C^i_j \\ &= -\frac{1}{2} \sum_i g^{ii} (\lambda^i_1 \lambda^j_2 - \lambda^i_2 \lambda^j_1)[-\partial_i g_{jj} g^{jj} D_{ji} + \partial_i \ln g^{jj} D_{ij}] \\ &= -\frac{1}{2} \sum_i g^{ii} (\lambda^i_1 \lambda^j_2 - \lambda^i_2 \lambda^j_1) \partial_i \ln g^{jj} (D_{ij} + D_{ji}). \end{aligned} \tag{4.14}$$

Thus, (4.10) follows from $(4.12)+(4.13)-(4.14)$. ■

Remark 4.4: From the first equations (4.9), it follows that: (i) $C^{ii}=0$, (ii) the diagonal components D_{ii} are not involved in the definition (4.8) of $\mathbf{C}$, (iii) if $\mathbf{D}$ is symmetric, $D_{ij}=D_{ji}$, then $C^{ij}+C^{ji}=0$ and $\mathbf{C}$ is skew-symmetric.

Remark 4.5: For $D_{ij}=\partial_i \Gamma_j$, due to (4.6), Eq. (4.10) becomes

$$\nabla_i C^i_j = \sum_i g^{ii} (\lambda^i_1 \lambda^j_2 - \lambda^i_2 \lambda^j_1)(\partial^2_i \Gamma_j - \Gamma_i \partial_i \Gamma_j), \tag{4.15}$$

and, due to the first equations (4.9), $C^{ij}+C^{ji}=0$. Hence, $\mathbf{C}$ is skew-symmetric and (4.15) gives the components of $\delta\mathbf{C}$. It follows that

$$\delta\mathbf{C}=0 \Leftrightarrow \sum_i g^{ii} (\lambda^i_1 \lambda^j_2 - \lambda^i_2 \lambda^j_1)(\partial^2_i \Gamma_j - \Gamma_i \partial_i \Gamma_j)=0. \tag{4.16}$$

Remark 4.6: For $\mathbf{K}_1=\mathbf{K}$ and $\mathbf{K}_2=\mathbf{G}$, the definition (4.8) and equations (4.9) become

$$\mathbf{C}=\mathbf{KD}-\mathbf{DK}, \quad C^{ij}=g^{ii}g^{jj}(\lambda_i-\lambda_j)D_{ij}, \quad C^i_j = g^{ii}(\lambda^i-\lambda^j)D_{ij},$$

and (4.10) reduces to

$$\nabla_i C^i_j = \sum_i g^{ii}(\lambda^i-\lambda^j)(\partial_i D_{ij} - \Gamma_i D_{ij} + \tfrac{1}{2}\partial_i \ln g^{jj}(D_{ji}-D_{ij})).$$

For $D_{ij}=\partial_i \Gamma_j$, because of (4.5), we have

$$\nabla_i C^i_j = \sum_i g^{ii}(\lambda^i-\lambda^j)(\partial^2_i \Gamma_j - \Gamma_i \partial_i \Gamma_j). \tag{4.17}$$

$\mathbf{C}$ is skew-symmetric and (4.17) gives the components of $\delta\mathbf{C}$. Thus,

$$\delta\mathbf{C}=0 \Leftrightarrow \sum_i g^{ii}(\lambda^i-\lambda^j)(\partial^2_i \Gamma_j - \Gamma_i \partial_i \Gamma_j)=0. \tag{4.18}$$

Remark 4.7: If the Killing tensor **K** in Proposition 4.1 has simple eigenvalues, $\lambda^i \neq \lambda^j$ for $i \neq j$, then (4.5) implies

$$\partial_i \Gamma_j = \partial_j \Gamma_i . \tag{4.19}$$

This is the case of a characteristic Killing tensor associated with the orthogonal separation. This proves

Proposition 4.8: Equation (4.19) holds for any separable orthogonal coordinate system.

This property has some interesting consequences. First, from (4.19) and (4.3) it follows that

Proposition 4.9: In any orthogonal separable coordinate system

$$\partial_i \partial_j \ln g^{ii} = \partial_i \partial_j \ln g^{jj}, \quad i \neq j.$$

A second consequence is concerned with the eigenvalues of a characteristic Killing tensor.

Proposition 4.10: For the eigenvalues (λ^i) of a characteristic Killing tensor of a Killing–Stäckel algebra the following equations hold:

$$\partial_i \partial_j \ln g^{ii} = -\frac{\partial_j \lambda^i \partial_i \lambda^j}{(\lambda^i - \lambda^j)^2}, \quad i \neq j, \tag{4.20}$$

Proof: For $\lambda^i \neq \lambda^j$, the first equations (4.1) can be written

$$\partial_j \ln g^{ii} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}.$$

If we apply the partial derivative ∂_i to this formula, and use again (4.1), then we get Eq. (4.20).■

A third consequence is concerned with the Robertson and the pre-Roberston conditions.

Proposition 4.11: For any orthogonal separable coordinate system $\underline{q} = (q^i)$ there are local functions $F(\underline{q})$ such that

$$\Gamma_i = \partial_i F.$$

The Robertson condition (3.3c) is equivalent to

$$\partial_i \partial_j F = 0, \quad i \neq j, \tag{4.21}$$

and the pre-Robertson condition (3.4) is equivalent to

$$\partial_j [\partial_i^2 F - \tfrac{1}{2}(\partial_i F)^2] = 0, \quad i \neq j. \tag{4.22}$$

Equation (4.21) means that the function F is a sum of functions depending on a single coordinate i.e., of functions constant on the leaves of the web: $F = \Sigma_i F_i(q^i)$. Equation (4.22) means that each function $\partial_i^2 F - 1/2(\partial_i F)^2$ is a function of the coordinate corresponding to the index only. A further interpretation of the pre-Robertson condition is expressed by the following

Proposition 4.12: The pre-Robertson condition is equivalent to

$$\partial_i Q_{ij} = 0, \quad i \neq j \ \text{ n.s.},$$

where

$$Q_{ij} = e^{-F} R_{ij} . \tag{4.23}$$

Proof: $\partial_i Q_{ij} = e^{-F}(\partial_i R_{ij} - \partial_i F R_{ij})$. ■

Remark 4.13: Let $\underline{q} = (q^i)$ and $\underline{q}' = (q^{i'})$ be two equivalent and equioriented orthogonal separable coordinate systems. Let us set

$$A_i^{i'} = \frac{\partial q^{i'}}{\partial q^i}, \quad A_{i'}^i = \frac{\partial q^i}{\partial q^{i'}}, \quad A = \det[A_i^{i'}] = \prod_i A_i^{i'},$$
$$G = \det[g^{ij}] = \prod_i g^{ii}, \quad G' = \det[g^{i'j'}] = \prod_{i'} g^{i'i'}. \tag{4.24}$$

Note that $A_i^{j'} = 0$ for $i \neq j$ and that $A > 0$. The link between functions F and F' corresponding to these coordinates is

$$F' = F - \ln A + \text{const.}$$

i.e.,

$$e^{-F'} = cAe^{-F}. \tag{4.25}$$

Indeed, the relationship between the contracted Christoffel symbols is

$$\Gamma_{i'} = A_{i'}^i (\Gamma_i - \partial_i \ln A_i^{i'}) = A_{i'}^i (\Gamma_i - \partial_i \ln A). \tag{4.26}$$

To prove (4.26) we observe that, since $A_i^{i'}$ is a function of q^i only,

$$\partial_i \ln A = \partial_i \ln \prod_j A_j^{j'} = \partial_i \ln A_i^{i'}.$$

Moreover, since $A > 0$, from (4.24) it follows that

$$\sqrt{G'} = A\sqrt{G},$$
$$\partial_{i'} \ln \sqrt{G'} = A_{i'}^i (\partial_i \ln A + \partial_i \ln \sqrt{G}), \tag{4.27}$$
$$\partial_{i'} \ln g^{i'i'} = A_{i'}^i (2\partial_i \ln A_i^{i'} + \partial_i \ln g^{ii}).$$

By (4.3) and (4.27) we get

$$\begin{aligned}\Gamma_{i'} &= \partial_{i'} \ln \sqrt{G'} - \partial_{i'} \ln g^{i'i'} \\ &= A_{i'}^i (\partial_i \ln A + \partial_i \ln \sqrt{G} - 2\partial_i \ln A_i^{i'} - \partial_i \ln g^{ii}) \\ &= A_{i'}^i (\Gamma_i + \partial_i \ln A - 2\partial_i \ln A_i^{i'}),\end{aligned}$$

which implies both equations (4.26). Finally, we observe that the object Q_{ij} is defined by (4.23) up to a multiplicative constant, since F is defined up to an additive constant. From (4.23), (4.25) and the first equation (4.27) it follows that

$$Q_{i'j'} = e^{-F'} R_{i'j'} = cAe^{-F} A_{i'}^i A_{j'}^j R_{ij} = cAA_{i'}^i A_{j'}^j Q_{ij}.$$

Thus,

$$\frac{1}{\sqrt{G'}} Q_{i'j'} = \frac{c}{\sqrt{G}} A_{i'}^i A_{j'}^j Q_{ij}.$$

V. COMMUTATION RELATIONS IN ORTHOGONAL COORDINATES

Proposition 5.1: If a symmetric tensor $\mathbf{K}=(K^{ij})$ *is diagonalized in orthogonal coordinates, then the corresponding pseudo-Laplacian assumes the form*

$$\Delta_{\mathbf{K}}\psi=A^i\partial_i^2\psi+B^i\partial_i\psi,\quad A^i=K^{ii}=\lambda^i g^{ii},\quad B^i=g^{ii}(\partial_i\lambda^i-\lambda^i\Gamma_i). \tag{5.1}$$

Proof: By definition (2.1) and formula (4.4),

$$\begin{aligned}\Delta_{\mathbf{K}}\psi&=\nabla_i(K^{ij}\partial_j\psi)\\&=\partial_i(K^{ij}\partial_j\psi)+\Gamma^i_{ih}K^{hj}\partial_j\psi\\&=\partial_i(K^{ii}\partial_i\psi)+\Gamma^i_{ih}K^{hh}\partial_h\psi\\&=K^{ii}\partial_i^2\psi+(\partial_hK^{hh}+\Gamma^i_{ih}K^{hh})\partial_h\psi\\&=\lambda^i g^{ii}\partial_i^2\psi+(\partial_h(\lambda^h g^{hh})-\lambda^h g^{hh}(\Gamma_h+\partial_h\ln g^{hh}))\partial_h\psi\\&=\lambda^i g^{ii}\partial_i^2\psi+g^{hh}(\partial_h\lambda^h-\lambda^h\Gamma_h)\partial_h\psi.\end{aligned}$$

■

Remark 5.2: For the ordinary Laplacian, $K^{ii}=g^{ii}$, $\lambda^i=1$, and $B^i=-g^{ii}\Gamma_i$, so that

$$\Delta\psi=g^{ii}[\partial_i^2\psi-\Gamma_i\partial_i\psi].$$

A second-order operator (2.12) assumes the form

$$\hat{H}_{\mathbf{K}}\psi=\frac{\hbar^2}{2}(A^i\partial_i^2\psi+B^i\partial_i\psi)+V_{\mathbf{K}}\psi=\frac{\hbar^2}{2}g^{ii}(\lambda^i\partial_i^2\psi+(\partial_i\lambda^i-\lambda^i\Gamma_i)\partial_i\psi)+V_{\mathbf{K}}\psi. \tag{5.2}$$

For a Killing tensor **K** Eqs. (4.1) hold, so that

$$\Delta_{\mathbf{K}}\psi=g^{ii}\lambda^i[\partial_i^2\psi-\Gamma_i\partial_i\psi].$$

For a basis $(\mathbf{K}_j)$ of a Killing–Stäckel algebra we have $\varphi^i_{(j)}=\lambda^i_j g^{ii}$ (Remark 3.6) and we find expressions (3.6) of the corresponding operators $\hat{H}_j$.

Proposition 5.3: The commutator of two second-order operators of the kind (5.2) has the following expression:

$$\begin{aligned}[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]\psi=&\frac{\hbar^4}{2}(A^i_1\partial_iA^j_2-A^i_2\partial_iA^j_1)\partial_i\partial_j^2\psi+\frac{\hbar^4}{4}(A^i_1\partial_i^2A^j_2-A^i_2\partial_i^2A^j_1+B^i_1\partial_iA^j_2-B^i_2\partial_iA^j_1)\partial_j^2\psi\\&+\frac{\hbar^4}{2}(A^i_1\partial_iB^j_2-A^i_2\partial_iB^j_1)\partial_i\partial_j\psi+\Bigg(\frac{\hbar^4}{4}(A^i_1\partial_i^2B^j_2+B^i_1\partial_iB^j_2-A^i_2\partial_i^2B^j_1-B^i_2\partial_iB^j_1)\\&-\hbar^2(A^j_1\partial_jV_{\mathbf{K}_2}-A^j_2\partial_jV_{\mathbf{K}_1})\Bigg)\partial_j\psi-\frac{\hbar^2}{2}(\Delta_{\mathbf{K}_1}V_{\mathbf{K}_2}-\Delta_{\mathbf{K}_2}V_{\mathbf{K}_1})\psi.\end{aligned} \tag{5.3}$$

Proof: For two second-order operators (2.2),

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=\left[\frac{1}{2}\hat{P}_{\mathbf{K}_1}+V_{\mathbf{K}_1},\frac{1}{2}\hat{P}_{\mathbf{K}_2}+V_{\mathbf{K}_2}\right]$$
$$=\frac{1}{4}[\hat{P}_{\mathbf{K}_1},\hat{P}_{\mathbf{K}_2}]+\frac{1}{2}[\hat{P}_{\mathbf{K}_1},V_{\mathbf{K}_2}]-\frac{1}{2}[\hat{P}_{\mathbf{K}_2},V_{\mathbf{K}_1}]$$
$$=\frac{\hbar^4}{4}[\Delta_{\mathbf{K}_1},\Delta_{\mathbf{K}_2}]-\frac{\hbar^2}{2}[\Delta_{\mathbf{K}_1},V_{\mathbf{K}_2}]+\frac{\hbar^2}{2}[\Delta_{\mathbf{K}_2},V_{\mathbf{K}_1}].$$

Since

$$\delta(f\mathbf{X})=\mathbf{X}\cdot\nabla f+f\,\delta\mathbf{X},$$

we have

$$[\Delta_{\mathbf{K}},V]\psi=\Delta_{\mathbf{K}}(V\psi)-V\Delta_{\mathbf{K}}\psi$$
$$=\delta(\mathbf{K}\nabla(V\psi))-V\Delta_{\mathbf{K}}\psi$$
$$=\delta((\mathbf{K}\nabla V)\psi+(\mathbf{K}\nabla\psi)V)-V\Delta_{\mathbf{K}}\psi$$
$$=\psi\Delta_{\mathbf{K}}V+2\mathbf{K}(\nabla\psi,\nabla V)$$
$$=\Delta_{\mathbf{K}}V\psi+2A^i\partial_iV\partial_i\psi.$$

Hence,

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]\psi=\frac{\hbar^4}{4}[\Delta_{\mathbf{K}_1},\Delta_{\mathbf{K}_2}]\psi+\frac{\hbar^2}{2}(\Delta_{\mathbf{K}_2}V_{\mathbf{K}_1}-\Delta_{\mathbf{K}_1}V_{\mathbf{K}_2})\psi+\hbar^2(\mathbf{K}_2\nabla V_{\mathbf{K}_1}-\mathbf{K}_1\nabla V_{\mathbf{K}_2})\cdot\nabla\psi, \tag{5.4}$$

Because of (5.1),

$$\Delta_{\mathbf{K}_1}\Delta_{\mathbf{K}_2}\psi=A_1^i\partial_i^2(A_2^j\partial_j^2\psi+B_2^j\partial_j\psi)+B_1^i\partial_i(A_2^j\partial_j^2\psi+B_2^j\partial_j\psi)$$
$$=A_1^i(\partial_i^2A_2^j\partial_j^2\psi+2\partial_iA_2^j\partial_i\partial_j^2\psi+A_2^j\partial_i^2\partial_j^2\psi+\partial_i^2B_2^j\partial_j\psi+2\partial_iB_2^j\partial_i\partial_j\psi+B_2^j\partial_i^2\partial_j\psi)$$
$$+B_1^i(\partial_iA_2^j\partial_j^2\psi+A_2^j\partial_i\partial_j^2\psi+\partial_iB_2^j\partial_j\psi+B_2^j\partial_i\partial_j\psi)$$
$$=A_1^iA_2^j\partial_i^2\partial_j^2\psi+(2A_1^i\partial_iA_2^j+B_1^iA_2^j+A_1^jB_2^i)\partial_i\partial_j^2\psi+(A_1^i\partial_i^2A_2^j+B_1^i\partial_iA_2^j)\partial_j^2\psi$$
$$+(2A_1^i\partial_iB_2^j+B_1^iB_2^j)\partial_i\partial_j\psi+(A_1^i\partial_i^2B_2^j+B_1^i\partial_iB_2^j)\partial_j\psi,$$

so that,

$$[\Delta_{\mathbf{K}_1},\Delta_{\mathbf{K}_2}]\psi=\Delta_{\mathbf{K}_1}\Delta_{\mathbf{K}_2}\psi-\Delta_{\mathbf{K}_2}\Delta_{\mathbf{K}_1}\psi$$
$$=2(A_1^i\partial_iA_2^j-A_2^i\partial_iA_1^j)\partial_i\partial_j^2\psi+(A_1^i\partial_i^2A_2^j-A_2^i\partial_i^2A_1^j+B_1^i\partial_iA_2^j-B_2^i\partial_iA_1^j)\partial_j^2\psi$$
$$+2(A_1^i\partial_iB_2^j-A_2^i\partial_iB_1^j)\partial_i\partial_j\psi+(A_1^i\partial_i^2B_2^j+B_1^i\partial_iB_2^j-A_2^i\partial_i^2B_1^j-B_2^i\partial_iB_1^j)\partial_j\psi.$$

Thus, from (5.4) we derive (5.3). ∎

Proposition 5.4: Let $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be symmetric tensors simultaneously diagonalized in orthogonal coordinates. Then,* $[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0$ *if and only if*

$$A_1^i\partial_iA_2^j-A_2^i\partial_iA_1^j=0\quad(i\ \text{ n.s.}), \tag{5.5a}$$

$$\sum_i (A_1^i\partial_i^2 A_2^j - A_2^i\partial_i^2 A_1^j + B_1^i\partial_i A_2^j - B_2^i\partial_i A_1^j) + 2(A_1^j\partial_j B_2^j - A_2^j\partial_j B_1^j) = 0, \tag{5.5b}$$

$$A_1^i\partial_i B_2^j - A_2^i\partial_i B_1^j + A_1^j\partial_j B_2^i - A_2^j\partial_j B_1^i = 0 \quad (i\neq j \;\; \text{n.s.}), \tag{5.5c}$$

$$\frac{\hbar^2}{4}\sum_i (A_1^i\partial_i^2 B_2^j + B_1^i\partial_i B_2^j - A_2^i\partial_i^2 B_1^j - B_2^i\partial_i B_1^j) - A_1^j\partial_j V_{\mathbf{K}_2} + A_2^j\partial_j V_{\mathbf{K}_1} = 0 \quad (j \;\; \text{n.s.}), \tag{5.5d}$$

$$\Delta_{\mathbf{K}_1} V_{\mathbf{K}_2} - \Delta_{\mathbf{K}_2} V_{\mathbf{K}_1} = 0. \tag{5.5e}$$

Proof: (i) Assume that (5.3) vanishes identically for all functions ψ. For $\psi=1$ we get (5.5e) and the last term in (5.3) disappears. For $\psi=q^j$ we get (5.5d), so that also the fourth term in (5.3) disappears. As a consequence, for $\psi=(q^j)^2$ we get (5.5b) and we reduce the vanishing of (5.3) to

$$\sum_{i,j} (A_1^i\partial_i A_2^j - A_2^i\partial_i A_1^j)\partial_i\partial_j^2\psi + \sum_{i,j\neq} (A_1^i\partial_i B_2^j - A_2^i\partial_i B_1^j)\partial_i\partial_j\psi = 0. \tag{5.6}$$

For $\psi=q^1q^2$, we have $\partial_i\partial_j\psi=\delta_j^1\delta_i^2+\delta_i^1\delta_j^2$, thus we get (5.5c) for distinct indices and moreover, (5.6) reduces to

$$\sum_{i,j} (A_1^i\partial_i A_2^j - A_2^i\partial_i A_1^j)\partial_i\partial_j^2\psi = 0. \tag{5.7}$$

Finally, for $\psi=q^1(q^2)^2$ we have $\partial_j^2\partial_i\psi=\partial_j^2(\delta_i^1(q^2)^2+2q^1q^2\delta_i^2)=2\delta_i^1\delta_j^2+4\delta_j^1\delta_j^2\delta_i^2 = 2\delta_i^1\delta_j^2$. Thus, we get (5.5a) for distinct indices (and no summation), so that (5.7) reduces to

$$\sum_j (A_1^j\partial_j A_2^j - A_2^j\partial_j A_1^j)\partial_j^3\psi = 0.$$

This shows that (5.5a) also holds for $i=j$. (ii) Conversely, assume that (5.5) hold. Then, due to (5.5a,d,e), Eq. (5.3) reduces to

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]\psi = \frac{\hbar^4}{4}(A_1^i\partial_i^2 A_2^j - A_2^i\partial_i^2 A_1^j + B_1^i\partial_i A_2^j - B_2^i\partial_i A_1^j)\partial_j^2\psi + \frac{\hbar^4}{2}(A_1^i\partial_i B_2^j - A_2^i\partial_i B_1^j)\partial_i\partial_j\psi$$

and, because of (5.5b), we obtain

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]\psi = \frac{\hbar^4}{2}\sum_{i,j\neq} (A_1^i\partial_i B_2^j - A_2^i\partial_i B_1^j)\partial_i\partial_j\psi.$$

But this last expression vanishes identically because of the skew-symmetry of Eq. (5.5c). ■

Remark 5.5: Since

$$\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\} = \{A_1^i p_i^2, A_2^j p_j^2\} = 2p_i p_j^2(A_1^i\partial_i A_2^j - A_2^i\partial_i A_1^j), \tag{5.8}$$

Eq. (5.5a) is equivalent to $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$. Thus,

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0 \Rightarrow \{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0. \tag{5.9}$$

Theorem 5.6: *Let* **K** *be a symmetric tensor diagonalized in orthogonal coordinates. Then the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}},\hat{H}]=0 \Leftrightarrow \begin{array}{c}\{P_{\mathbf{K}},P_{\mathbf{G}}\}=0 \quad (\mathbf{K}\ \text{Killing tensor}) \\ \frac{\hbar^2}{4}\delta\mathbf{C}+\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}=0\end{array} \Leftrightarrow \{H_{\mathbf{K}},H\}=-\frac{\hbar^2}{4}P_{\delta C}, \tag{5.10}$$

where

$$C=\mathbf{KD}-\mathbf{DK}, \quad \mathbf{D}=(D_{ij})=(\partial_i\Gamma_j).$$

Proof: We use (5.5) of Proposition 5.4 for the case

$$\begin{array}{c}\mathbf{K}_1=\mathbf{K} \\ \mathbf{K}_2=\mathbf{G}\end{array} \Leftrightarrow \begin{array}{c}\lambda_1^i=\lambda^i \\ \lambda_2^i=1\end{array} \Leftrightarrow \begin{array}{ll}A_1^i=\lambda^i g^{ii}, & B_1^i=g^{ii}(\partial_i\lambda^i-\lambda^i\Gamma_i) \\ A_2^i=g^{ii}, & B_2^i=-g^{ii}\Gamma_i\end{array}.$$

Assume $[\hat{H}_{\mathbf{K}},\hat{H}_{\mathbf{G}}]=0$. From (5.9) it follows that $\mathbf{K}$ is a Killing tensor. Then we use (4.1) and in (5.5) we consider

$$B_1^i=-g^{ii}\lambda^i\Gamma_i, \quad B_2^i=-g^{ii}\Gamma_i, \tag{5.11}$$

and

$$\begin{aligned}
&\partial_i A_1^j=\lambda^i\partial_i g^{jj}, \quad \partial_i B_1^j=-\partial_i(g^{jj}\lambda^j\Gamma_j)=-\lambda^i\partial_i g^{jj}\Gamma_j-g^{jj}\lambda^j\partial_i\Gamma_j,\\
&\partial_i A_2^j=\partial_i g^{jj}, \quad \partial_i B_2^j=-\partial_i(g^{jj}\Gamma_j)=-\partial_i g^{jj}\Gamma_j-g^{jj}\partial_i\Gamma_j,\\
&\partial_i^2 A_1^j=\lambda^i\partial_i^2 g^{jj}, \quad \partial_i^2 B_1^j=-\lambda^i\partial_i^2 g^{jj}\Gamma_j-2\lambda^i\partial_i g^{jj}\partial_i\Gamma_j-g^{jj}\lambda^j\partial_i^2\Gamma_j,\\
&\partial_i^2 A_2^j=\partial_i^2 g^{jj}, \quad \partial_i^2 B_2^j=-\partial_i^2 g^{jj}\Gamma_j-2\partial_i g^{jj}\partial_i\Gamma_j-g^{jj}\partial_i^2\Gamma_j,
\end{aligned} \tag{5.12}$$

Equations (5.5a) and (5.5b) are then identically satisfied, while the remaining three equations become

$$\begin{aligned}
&(\lambda^i-\lambda^j)(\partial_i\Gamma_j-\partial_j\Gamma_i)=0 \quad (i\neq j\ \text{n.s.}),\\
&\frac{\hbar^2}{4}\sum_i g^{ii}(\lambda^i-\lambda^j)(\partial_i^2\Gamma_j-\Gamma_i\partial_i\Gamma_j)+\lambda^j\partial_j V-\partial_j V_{\mathbf{K}}=0 \quad (j\ \text{n.s.}),\\
&\delta(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}})=0.
\end{aligned} \tag{5.13}$$

Due to (4.5), the first equation is identically satisfied. According to Remark 4.6 and Eq. (4.17), the second equation (5.13) is equivalent to

$$\frac{\hbar^2}{4}\delta\mathbf{C}+\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}=0,$$

where $\mathbf{C}$ is skew-symmetric. Since $\delta^2\mathbf{C}=0$, the last equation (5.13) is a consequence of the second equation (5.13). The above-given reasoning is reversible, and the first equivalence (5.10) is proved. The second equivalence follows from last equation (2.3). ■

Remark 5.7: The comparison between Theorem 2.2, proved by using the Carter formula (2.8) without any special assumption on $\mathbf{K}$, and Theorem 5.6, proved under the assumption that $\mathbf{K}$ is diagonalized in orthogonal coordinates, shows that for such a Killing tensor the following equation holds:

$$\delta(\mathbf{KD}-\mathbf{DK})=\tfrac{2}{3}\delta(\mathbf{KR}-\mathbf{RK}), \tag{5.14}$$

where $\mathbf{D}=(\partial_i\Gamma_j)$. Note that the components D_{ii} are not involved in the definition of $\mathbf{C}=\mathbf{KD}-\mathbf{DK}$ (Remark 4.4). This is in accordance with formula (3.5), which holds in separable orthogonal coordinates.

Now we apply Proposition 5.4 to the case of two Killing tensors.

Theorem 5.8: *Let* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be Killing tensors simultaneously diagonalized in orthogonal coordinates. Then* $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$ *and the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0 \Leftrightarrow \frac{\hbar^2}{4}\,\delta\mathbf{C}+\mathbf{K}_1\nabla V_{\mathbf{K}_2}-\mathbf{K}_2\nabla V_{\mathbf{K}_1}=0 \Leftrightarrow \{H_{\mathbf{K}_1},H_{\mathbf{K}_2}\}=-\frac{\hbar^2}{4}P_{\delta\mathbf{C}}, \tag{5.15}$$

where

$$\mathbf{C}=\mathbf{K}_1\mathbf{DK}_2-\mathbf{K}_2\mathbf{DK}_1, \quad \mathbf{D}=(\partial_i\Gamma_j). \tag{5.16}$$

Proof: The components of $\mathbf{C}$ defined in (5.16) are [recall (4.8) and (4.9)]

$$C^{ij}=g^{ii}g^{jj}(\lambda_1^i\lambda_2^j-\lambda_2^i\lambda_1^j)\partial_i\Gamma_j, \quad C^i_j=C^{i}_{\cdot j}=g^{ii}(\lambda_1^i\lambda_2^j-\lambda_2^i\lambda_1^j)\partial_i\Gamma_j.$$

The involutivity condition $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$ follows from (5.8), (5.1), and (4.1). We use (5.5). According to (5.1), (5.11), and (5.12), for a Killing tensor $\mathbf{K}_I$ $(I=1,2)$ we have

$$\begin{aligned}
&A_I^i=\lambda_I^i g^{ii}, \quad B_I^i=-A_I^i\Gamma_i,\\
&\partial_i A_I^j=A_I^i\partial_i g^{jj}=\lambda_I^i\partial_i g^{jj},\\
&\partial_i^2 A_I^j=A_I^i\partial_i^2 g^{jj}=\lambda_I^i\partial_i^2 g^{jj}\\
&\partial_i B_I^j=-\partial_i(A_I^j\Gamma_j)=-\partial_i A_I^j\Gamma_j-A_I^j\partial_i\Gamma_j,\\
&\partial_i^2 B_I^j=-\partial_i^2 A_I^j\Gamma_j-2\,\partial_i A_I^j\partial_i\Gamma_j-A_I^j\partial_i^2\Gamma_j.
\end{aligned} \tag{5.17}$$

Because of the first two equations (5.17), Eq. (5.5a) is identically satisfied and the sum of the first two terms in Eq. (5.5b) vanishes, so that this equation reduces to

$$\sum_i \Gamma_i[A_1^i\partial_i A_2^j-A_2^i\partial_i A_1^j]+2[A_1^j\partial_j(A_2^j\Gamma_j)-A_2^j\partial_j(A_1^j\Gamma_j)]=0.$$

But all the terms in this sum vanish because of (5.5a). Thus, also (5.5b) is identically satisfied. Equation (5.5c) becomes

$$A_1^i\partial_i(A_2^j\Gamma_j)-A_2^i\partial_i(A_1^j\Gamma_j)+A_1^j\partial_j(A_2^i\Gamma_i)-A_2^j\partial_j(A_1^i\Gamma_i)=0.$$

Because of (5.5a) it reduces to

$$(A_1^iA_2^j-A_2^iA_1^j)(\partial_i\Gamma_j-\partial_j\Gamma_i)=0 \quad (i\neq j,\ \text{n.s.}),$$

that is (up to a factor $g^{ii}g^{jj}$) to (4.6), which is identically satisfied. Due to the last two equations (5.17), Eq. (5.5d) becomes

$$\frac{\hbar^2}{4}\sum_i [A_1^i\partial_i^2(A_2^j\Gamma_j)-A_1^i\Gamma_i\partial_i(A_2^j\Gamma_j)-A_2^i\partial_i^2(A_1^j\Gamma_j)+A_2^i\Gamma_i\partial_i(A_1^j\Gamma_j)]+A_1^j\partial_j V_{\mathbf{K}_2}-A_2^j\partial_j V_{\mathbf{K}_1}=0,$$

thus,

$$\frac{\hbar^2}{4}\sum_i [(A_1^iA_2^j-A_2^iA_1^j)\partial_i^2\Gamma_j]+\frac{\hbar^2}{4}\sum_i [A_1^i\Gamma_j\partial_i^2A_2^j-A_2^i\Gamma_j\partial_i^2A_1^j+2A_1^i\partial_iA_2^j\partial_i\Gamma_j-2A_2^i\partial_iA_1^j\partial_i\Gamma_j]$$

$$-\frac{\hbar^2}{4}\sum_i \Gamma_i[A_1^i\partial_i(A_2^i\Gamma_j)-A_2^i\partial_i(A_1^j\Gamma_j)]+A_1^j\partial_jV_{\mathbf{K}_2}-A_2^j\partial_jV_{\mathbf{K}_1}=0.$$

Because of the second equation (5.17) and (5.5a), the second sum vanishes identically and this equation reduces to

$$\frac{\hbar^2}{4}\sum_i [(A_1^iA_2^j-A_2^iA_1^j)\partial_i^2\Gamma_j]-\frac{\hbar^2}{4}\sum_i [\Gamma_i(A_1^iA_2^j-A_2^iA_1^j)\partial_i\Gamma_j]+A_1^j\partial_jV_{\mathbf{K}_2}-A_2^j\partial_jV_{\mathbf{K}_1}=0,$$

which is equivalent to

$$\frac{\hbar^2}{4}\sum_i [g^{ii}(\lambda_1^i\lambda_2^j-\lambda_2^i\lambda_1^j)(\partial_i^2\Gamma_j-\Gamma_i\partial_i\Gamma_j)]+\lambda_1^j\partial_jV_{\mathbf{K}_2}-\lambda_2^j\partial_jV_{\mathbf{K}_1}=0. \tag{5.18}$$

Due to (4.15) this equation is equivalent to the second equation in (5.15). So, the first commutation relation (5.15) is equivalent to the second equation (5.15) plus the last equation (5.5). However, **C** is skew-symmetric (Remark 4.5), so that the second equation (5.15) implies equation $\delta(\mathbf{K}_1\nabla V_{\mathbf{K}_2}-\mathbf{K}_2\nabla V_{\mathbf{K}_1})=0$, that is (5.5e). This proves the first equivalence (5.15). The second equivalence follows from the first equation (2.3), which now reads $\{H_{\mathbf{K}_1},H_{\mathbf{K}_2}\}=P(\mathbf{K}_1\nabla V_{\mathbf{K}_2}-\mathbf{K}_2\nabla V_{\mathbf{K}_1})$.

Proposition 5.9: Let $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be Killing tensors simultaneously diagonalized in orthogonal coordinates. Then*

$$[\hat{H}_{\mathbf{K}_1},\hat{H}]=0,\quad [\hat{H}_{\mathbf{K}_2},\hat{H}]=0\Rightarrow[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0. \tag{5.19}$$

Proof: Since $\{P_{\mathbf{K}_1},P_{\mathbf{G}}\}=0$, due to Theorem 5.6 the first two conditions (5.19) are equivalent to

$$\frac{\hbar^2}{4}\delta\mathbf{C}+\mathbf{K}_I\nabla V-\nabla V_{\mathbf{K}_I}=0,\quad I=1,2.$$

Because of (4.17), Remark 4.6, these equations are equivalent to

$$\begin{aligned}&\frac{\hbar^2}{4}\sum_i g^{ii}(\lambda_1^i-\lambda_1^j)(\partial_i^2\Gamma_j-\Gamma_i\partial_i\Gamma_j)+\lambda_1^j\partial_jV-\partial_jV_{\mathbf{K}_1}=0,\\ &\frac{\hbar^2}{4}\sum_i g^{ii}(\lambda_2^i-\lambda_2^j)(\partial_i^2\Gamma_j-\Gamma_i\partial_i\Gamma_j)+\lambda_2^j\partial_jV-\partial_jV_{\mathbf{K}_2}=0.\end{aligned} \tag{5.20}$$

As we have done above, if we multiply the first equation by $\lambda_2^j\neq 0$, the second one by $\lambda_1^j\neq 0$ and subtract the two resulting equations, then we get (5.18), which is equivalent to the first equation (5.15). For $\lambda_2^j=0$, Eq. (5.18) follows from the first equation (5.20) multiplied by $\lambda_1^j\neq 0$. For $\lambda_1^j=\lambda_2^j=0$, (5.18) is obviously satisfied. ∎

Now we are able to prove Theorem 3.1 by applying the preceding results to the space $\mathcal{H}=(\mathcal{K},V)$ of the first integrals in involution associated with the orthogonal separation of the Hamilton–Jacobi equation (Sec. III). First, we prove the equivalence of the first three conditions (3.3).

Proposition 5.10: Let $\mathcal{H}=(\mathcal{K},V)$ be the space of quadratic first integrals in involution associated with the orthogonal separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent:

$$[\hat{H}_{\mathbf{K}},\hat{H}]=0, \quad \forall \mathbf{K}\in\mathcal{K},$$

$$\delta(\mathbf{KR}-\mathbf{RK})=0, \quad \forall \mathbf{K}\in\mathcal{K}, \tag{5.21}$$

$$\partial_i R_{ij}-\Gamma_i R_{ij}=0, \quad (i\neq j \text{ n.s.}).$$

Proof: Let us use Theorem 5.6. Since $\{H_{\mathbf{K}},H\}=0$, the first equation (5.21), coinciding with the first equation (5.10), is equivalent to $\delta\mathbf{C}=0$ for $\mathbf{C}=\mathbf{KD}-\mathbf{DK}$ and $D_{ij}=\partial_i\Gamma_j$. However, since the components D_{ii} are not involved in this definition of $\mathbf{C}$ (Remark 4.4), we can replace $\mathbf{D}$ with $\frac{2}{3}\mathbf{R}$, in agreement with (5.14). This proves the equivalence of the first two conditions (5.21). From the equivalence (4.18) it follows that the coordinate expression of the second equation (5.21) is

$$\sum_i g^{ii}(\lambda^i-\lambda^j)(\partial_i R_{ij}-\Gamma_i R_{ij})=0, \tag{5.22}$$

where only the nondiagonal covariant components of $\mathbf{R}$ are involved. If we introduce the vectors

$$\mathbf{X}_j=(X_j^i)=(g^{ii}(\partial_i R_{ij}-\Gamma_i R_{ij})), \quad \mathbf{Y}_j=(Y_j^i)=(\lambda^i-\lambda^j),$$

then equation (5.22) can be written

$$\mathbf{X}_j\cdot\mathbf{Y}_j=0. \tag{5.23}$$

Let us consider a basis $(\mathbf{K}_a)=(K_a^{ii})=(\lambda_a^i g^{ii})$ of $\mathcal{K}$, $a=1,...,n$, with $\mathbf{K}_n=\mathbf{G}$. We have $\det[\lambda_a^i]\neq 0$ and $\lambda_n^i=1$. Let us chose a value of the index j, say $j=1$. Then the $n-1$ vectors $\mathbf{Y}_{1a}=(\lambda_a^i-\lambda_a^1)$, $a=1,2,...,n-1$, are independent vectors in the $(n-1)$-space Π_1 orthogonal to the vector (1, 0,..., 0). Indeed, the rank of the $n\times(n-1)$ matrix $[\lambda_a^i-\lambda_a^1]$ is maximal. According to the second equation (5.21), Eq. (5.23) must be satisfied by all these vectors:

$$\mathbf{X}_1\cdot\mathbf{Y}_{1a}=0.$$

This means that $\mathbf{X}_1$ is orthogonal to Π_1, i.e., that $X_1^i=0$ for $i\neq 1$. In a similar way we prove that $X_j^i=0$ for $j\neq i$. Thus, the second equation (5.21) implies the third one. The converse is obvious. ■

Proposition 5.11: Let $\mathcal{H}=(\mathcal{K},V)$ be the space of quadratic first integrals in involution associated with the orthogonal separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent,

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2,\in\mathcal{K},$$

$$\delta(\mathbf{K}_1\mathbf{R}\mathbf{K}_2-\mathbf{K}_2\mathbf{R}\mathbf{K}_1)=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2,\in\mathcal{K}, \tag{5.24}$$

$$\partial_i R_{ij}-\Gamma_i R_{ij}=0, \quad i\neq j.$$

Proof: The first condition is equivalent to $\delta\mathbf{C}=0$, because of the second equivalence (5.15) (Theorem 5.8), with $\mathbf{C}$ defined in (5.16). However, in definition (5.16) $\mathbf{D}$ can be replaced by $\mathbf{R}$, due to (5.14) and Remark 4.4(ii). Thus, the first two conditions (5.24) are equivalent. The second equation (5.24) implies the second equation (5.21), since $\mathbf{G}\in\mathcal{K}$, and the last equation (5.24) because of Proposition 5.10. The last condition (5.24) implies the second condition (5.24) because of (4.16). ■

The last condition (5.24) appears also in (5.21). Thus, all the conditions (5.24) and (5.21) are equivalent. This proves Theorem 3.1.

VI. SYMMETRY OPERATORS ASSOCIATED WITH THE GENERAL SEPARATION OF THE HAMILTON–JACOBI EQUATION

A *separable Killing algebra*[1] is a pair $(D,\mathcal{K})$ where D is an r-dimensional linear space of commuting Killing vectors and $\mathcal{K}$ is a D-invariant $(n-r)$-dimensional linear space of Killing two-tensors with $m=n-r$ normal eigenvectors orthogonal to D. These eigenvectors are called *essential eigenvectors*. The eigenvalues of a $\mathbf{K}\in\mathcal{K}$ corresponding to essential eigenvectors are called *essential eigenvalues*. It can be proved that: (i) D is normal, i.e., the distribution $\Delta^{\perp}$ orthogonal to the vectors of D is completely integrable, (ii) $\mathcal{K}$ contains the metric tensor $\mathbf{G}$ and Killing tensors with distinct essential eigenvalues (called *characteristic Killing tensors*); (iii) all functions $P_{\mathbf{X}}$ and $P_{\mathbf{K}}$, with $\mathbf{X}\in D$ and $\mathbf{K}\in\mathcal{K}$ are in involution; (iv) there exist *standard coordinates* (q^a,q^α) such that dq^a are eigenforms of $\mathcal{K}$ corresponding to the essential eigenvectors and ∂_α form a local basis of D, so that (q^α) are ignorable; (v) these coordinates are separable for the geodesic flow; (vi) in these coordinates all elements of $\mathcal{K}$ assume the *standard form*

$$\mathbf{K}=K^{aa}\partial_a\otimes\partial_a+K^{\alpha\beta}\partial_\alpha\otimes\partial_\beta=\lambda^a g^{aa}\partial_a\otimes\partial_a+K^{\alpha\beta}\partial_\alpha\otimes\partial_\beta, \tag{6.1}$$

where λ^a are the essential eigenvalues of $\mathbf{K}$ and the coordinates (q^α) are ignorable; (vii) the natural Hamiltonian $H=\frac{1}{2}P_{\mathbf{G}}+V$ is separable if and only if there exists a separable Killing algebra such that $DV=0$ and the *characteristic equation* $d(\mathbf{K}dV)=0$ is satisfied for a single characteristic Killing tensor of $\mathcal{K}$. It follows that (viii) the characteristic equation is satisfied for all $\mathbf{K}\in\mathcal{K}$ and that there are local D-invariant functions $V_{\mathbf{K}}$ such that $dV_{\mathbf{K}}=\mathbf{K}dV$, i.e.,

$$\nabla V_{\mathbf{K}}=\mathbf{K}\nabla V,\quad DV_{\mathbf{K}}=0;$$

(ix) the functions

$$P_{\mathbf{X}},\quad \mathbf{X}\in D,$$

$$H_{\mathbf{K}}=\tfrac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}},\quad \mathbf{K}\in\mathcal{K}$$

are first integrals in involution. We denote by

$$\mathcal{H}=(\mathcal{K},V)$$

the m-dimensional linear space of the quadratic first integrals $H_{\mathbf{K}}$.

For the first- and second-order operators corresponding to these first integrals in involution the commutation relations

$$[\hat{P}_{\mathbf{X}_1},\hat{P}_{\mathbf{X}_2}]=0,\quad [\hat{P}_{\mathbf{X}},\hat{H}_{\mathbf{K}}]=0,\quad \forall\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}\in D,\quad \forall\mathbf{K}\in\mathcal{K},$$

hold. This follows from the fact that $\mathbf{X}\in D$ are commuting Killing vectors and the elements of $\mathcal{H}$ are D-invariant. However, in general the operators $\hat{H}_{\mathbf{K}}$ do not commute one other. Indeed, we have a theorem similar to Theorem 3.1,

Theorem 6.1: *Let $\mathcal{H}=(\mathcal{K},V)$ be the space of quadratic first integrals in involution associated with the separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}},\hat{H}]=0,\quad \forall\mathbf{K}\in\mathcal{K}, \tag{6.2a}$$

$$\delta(\mathbf{KR}-\mathbf{RK})=0,\quad \forall\mathbf{K}\in\mathcal{K}, \tag{6.2b}$$

$$\partial_a R_{ab}-\Gamma_a R_{ab}=0,\quad a\neq b\ \ \text{n.s.}, \tag{6.2c}$$

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0,\quad \forall\mathbf{K}_1,\mathbf{K}_2,\in\mathcal{K}, \tag{6.2d}$$

$$\delta(\mathbf{K}_1\mathbf{R}\mathbf{K}_2-\mathbf{K}_2\mathbf{R}\mathbf{K}_1)=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2,\in \mathcal{K}, \tag{6.2e}$$

where $R_{ab}=\mathbf{R}(\partial_a,\partial_b)$ *are the essential covariant components of the Ricci tensor* $\mathbf{R}$ (*corresponding to essential separable coordinates* (q^a)) *and* $\Gamma_a=g^{ij}\Gamma_{ij,a}$ *are the essential contracted Christoffel symbols.*

The proof of this theorem will be given in Sec. VIII.

Remark 6.2: Formulas (6.2) are formally identical to formulas (3.3) concerning the orthogonal separation, and a remark similar to Remark 3.2 also holds in the present case. The only difference is that now the coordinate expression of the pre-Robertson condition (6.2c) involves only the essential components R_{ab} of the Ricci tensor. In standard separable coordinates the Ricci tensor $\mathbf{R}$ assume the form (cf. Ref. 1, Sec. VI)

$$\mathbf{R}=R^{ab}\partial_a\otimes\partial_b+R^{\alpha\beta}\partial_\alpha\otimes\partial_\beta, \tag{6.3}$$

and moreover,

$$\partial_a\Gamma_b=\tfrac{2}{3}R_{ab}, \quad a\neq b. \tag{6.4}$$

It assumes the standard form, i.e.,

$$R_{ab}=0, \quad a\neq b, \tag{6.5}$$

if and only if the Schrödinger equation is separable in the reduced sense (Robertson condition). The Robertson condition (6.5) obviously implies the pre-Robertson condition (6.3). Hence,

Theorem 6.3: *If the Schrödinger equation associated with a separable Hamiltonian is reductively separable, then all operators* $\hat{P}_{\mathbf{X}}$ *and* $\hat{H}_{\mathbf{K}}$ *corresponding to the linear and quadratic first integrals in involution commute.*

In particular they commute with the Schrödinger operator $\hat{H}=\hat{H}_{\mathbf{G}}$. The Robertson and pre-Robertson conditions are obviously satisfied for $\mathbf{R}=\kappa\mathbf{G}$. Hence,

Theorem 6.4: *On Einstein manifolds all operators* $\hat{P}_{\mathbf{X}}$ *and* $\hat{H}_{\mathbf{K}}$ *corresponding to the first integrals in involution of a separable Hamiltonian system commute.*

Remark 6.5: An algebraic form of the Robertson condition is expressed by the commutability of the Ricci tensor $\mathbf{R}$ with a characteristic tensor (thus, with all the Killing tensors) $\mathbf{K}\in\mathcal{K}$, interpreted as linear operators, when restricted to the distribution $\Delta^\perp$ orthogonal to D,

$$(\mathbf{KR}-\mathbf{RK})|\Delta^\perp=0, \quad \forall \mathbf{K}\in\mathcal{K}. \tag{6.6}$$

Indeed, this distribution is invariant with respect to these linear operators. If we denote by $\mathbf{R}'$ and $\mathbf{K}'$ the restrictions to $\Delta^\perp$, cf. (7.1), then (6.6) is equivalent to

$$\mathbf{K}'\mathbf{R}-\mathbf{R}\mathbf{K}'=0, \tag{6.7}$$

or to $\mathbf{K}\mathbf{R}'-\mathbf{R}'\mathbf{K}=\mathbf{K}'\mathbf{R}'-\mathbf{R}'\mathbf{K}'=0$. Condition (6.7) obviously implies

$$\delta(\mathbf{K}'\mathbf{R}-\mathbf{R}\mathbf{K}')=0. \tag{6.8}$$

As we shall see in Sec. VIII, the fact that (6.8) is equivalent to (6.2b) is remarkable.

Remark 6.6: In standard separable coordinates the components of the elements of $\mathcal{K}$ and the potential functions assume the form

$$g^{aa}=\varphi^a_{(m)}, \quad g^{\alpha\beta}=\phi^{\alpha\beta}_a(q^a)\varphi^a_{(m)}, \quad V=\phi_a(q^a)\varphi^a_{(m)},$$

$$K^{aa}_b=\varphi^a_{(b)}, \quad K^{\alpha\beta}_b=\phi^{\alpha\beta}_a(q^a)\varphi^a_{(b)}, \quad V_{K_b}=\phi_a(q^a)\varphi^a_{(b)},$$

where $(\mathbf{K}_b)$ is a local basis of $\mathcal{K}$, with $\mathbf{K}_m=\mathbf{G}$. Then, a local basis of $\mathcal{H}$ is

$$H_b = \tfrac{1}{2}\varphi^a_{(b)}(p_a^2 + \phi_a^{\alpha\beta} p_\alpha p_\beta + 2\phi_a).$$

The corresponding operators assume the form (see Remark 8.2 and Ref. 1, Sec. V).

$$\hat{H}_b \tilde{\psi} = -\frac{\hbar^2}{2}\varphi^a_{(b)}\left(\partial_a^2\tilde{\psi} - \Gamma_a \partial_a \tilde{\psi} + \left(\phi_a^{\alpha\beta}\kappa_\alpha\kappa_\beta - \frac{2}{\hbar^2}\phi_a\right)\tilde{\psi}\right),$$

where $\psi = \tilde{\psi}\cdot \Pi_\alpha e^{\kappa_\alpha q^\alpha}$, $\tilde{\psi} = \Pi_a \psi_a(q^a)$. The Robertson condition in standard separable coordinates is equivalent to $\partial_b \Gamma_a = 0$ for $a \neq b$. This means that $\Gamma_a = \Gamma_a(q^a)$.

VII. KILLING TENSORS IN STANDARD FORM

In the next section we shall analyze the commutation relations of the second-order operators assuming that all the tensors **K** involved, including the metric tensor **G**, are simultaneously in standard form (6.1) with respect to a given standard coordinate system $(q^i) = (q^a, q^\alpha)$. We shall use the decomposition

$$\mathbf{K} = \mathbf{K}' + \mathbf{K}'',$$

$$\mathbf{K}' = K^{aa}\partial_a \otimes \partial_a = \lambda^a g^{aa} \partial_a \otimes \partial_a, \tag{7.1}$$

$$\mathbf{K}'' = K^{\alpha\beta}\partial_\alpha \otimes \partial_\beta.$$

In analogy with Sec. IV, in this section we state some general properties concerning Killing tensors. For a Killing tensor in standard form the following equations hold:

$$\begin{aligned} &\partial_a \lambda^b = (\lambda^a - \lambda^b)\partial_a \ln g^{bb} \\ &\partial_a K^{\alpha\beta} = \lambda^a \partial_a g^{\alpha\beta} \\ &\partial_a \lambda^a = 0 \\ &\partial_a(\lambda^b g^{bb}) = \lambda^a \partial_a g^{bb} \qquad (a,b \ \text{ n.s.}). \\ &\partial_a^2(\lambda^b g^{bb}) = \lambda^a \partial_a^2 g^{bb} \\ &\partial_a^2 K^{\alpha\beta} = \lambda^a \partial_a^2 g^{\alpha\beta} \end{aligned} \tag{7.2}$$

We call the two first equations (7.2) the *extended Eisenhart–Killing equations*. They characterize the Killing tensors in standard form and imply the remaining equations.

Proposition 7.1: If (q^a, q^α) *are standard coordinates in which a Killing tensor* **K** *assumes the standard form (6.1), then*

$$(\lambda^a - \lambda^b)(\partial_a \Gamma_b - \partial_b \Gamma_a) = 0 \quad (a,b \ \text{ n.s.}). \tag{7.3}$$

Proof: The proof follows the same pattern of the proof of Proposition 4.1. The only difference is that (4.3) is replaced by (cf. Ref. 1, Sec. VI)

$$\Gamma_a = \tfrac{1}{2}\partial_a \sum_c \ln g^{cc} - \partial_a \ln g^{aa} + \tfrac{1}{2}\partial_a \ln \det[g^{\alpha\beta}],$$

but the last term does not give any contribution to the difference $\partial_a \Gamma_b - \partial_b \Gamma_a$. ■

In a similar way we can prove

Proposition 7.2: Let $\mathbf{K}_I$, $I = 1,2$, *be two Killing tensors in standard form (6.1). Then*

$$(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)(\partial_a\Gamma_b-\partial_b\Gamma_a)=0 \quad (a\neq b \ \text{ n.s.}). \tag{7.4}$$

Proposition 7.3: Let $\mathbf{K}_I$, $I=1,2$, *be two Killing tensors in standard form (6.1). If*

$$\mathbf{C}=\mathbf{K}_1'\mathbf{D}\mathbf{K}_2'-\mathbf{K}_2'\mathbf{D}\mathbf{K}_1', \quad C^{ij}=(K_1')^{ih}D_{hk}(K_2')^{kj}-(K_2')^{ih}D_{hk}(K_1')^{kj}, \tag{7.5}$$

where $\mathbf{D}=(D_{ij})$ *is a geometrical object, then*

$$\begin{aligned} &C^{ab}=g^{aa}g^{bb}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)D_{ab}, \quad C^{a\alpha}=C^{\alpha a}=C^{\alpha\beta}=0,\\ &C_b^a=C_{\cdot b}^a=g^{aa}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)D_{ab}, \quad C_\alpha^a=C_a^\alpha=C_\alpha^\beta=0, \end{aligned} \tag{7.6}$$

and

$$\nabla_i C_\alpha^i=0, \quad \nabla_i C_b^i=\sum_a g^{aa}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)(\partial_a D_{ab}-\Gamma_a D_{ab}+\tfrac{1}{2}\partial_a \ln g^{bb}(D_{ba}-D_{ab})). \tag{7.7}$$

Proof: Equations (7.6) are a direct consequence of definitions (7.5) and (7.1). In standard coordinates (cf. Ref. 1, Sec. VI)

$$\Gamma_{i\alpha}^i=0, \quad \Gamma_{ia}^i=-\tfrac{1}{2}\partial_a(\ln\det[g^{ij}])=-\partial_a \ln g^{aa}-\Gamma_a, \tag{7.8}$$

and formula (4.11) reduces to

$$\nabla_i C_j^i=\partial_a C_j^a+\Gamma_{ia}^i C_j^a-\Gamma_{ij}^h C_h^i.$$

It follows that $\nabla_i C_\alpha^i=0$ and

$$\nabla_i C_b^i=\partial_a C_b^a-(\Gamma_a+\partial_a \ln g^{aa})C_b^a-\Gamma_{ab}^c C_c^a.$$

The development of this last expression follows the same pattern of the proof of Proposition 4.3.■

Remark 7.4: From (7.6) it follows that: (i) $C^{ii}=0$, (ii) only the nondiagonal components D_{ab}, $a\neq b$, are involved in the definition (7.5) of $\mathbf{C}$, (iii) if the essential components of D are symmetric, $D_{ab}=D_{ba}$, then $\mathbf{C}$ is skew-symmetric.

Remark 7.5: Let us apply Proposition 7.3 to the cases $D_{ij}=\partial_i\Gamma_j$. In standard coordinates $\Gamma_\alpha=0$ and $\partial_\alpha\Gamma_a=0$. Thus, $D_{\alpha\beta}=D_{a\alpha}=D_{\alpha a}=0$ and $\mathbf{C}$ defined in (7.5) is equal to

$$\mathbf{C}=\mathbf{K}_1\mathbf{D}\mathbf{K}_2-\mathbf{K}_2\mathbf{D}\mathbf{K}_1, \quad \mathbf{D}=(\partial_i\Gamma_j).$$

Equations (7.6) hold with D_{ab} replaced by $\partial_a\Gamma_b$,

$$C^{ab}=g^{aa}g^{bb}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)\partial_a\Gamma_b, \quad C_b^a=g^{aa}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)\partial_a\Gamma_b, \tag{7.9}$$

and, due to (7.4), equations (7.7) become

$$\nabla_i C_\alpha^i=0, \quad \nabla_i C_b^i=\sum_a g^{aa}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)(\partial_a^2\Gamma_b-\Gamma_a\partial_a\Gamma_b). \tag{7.10}$$

From (7.4) and (7.9) it follows that $C^{ij}+C^{ji}=0$. Hence, $\mathbf{C}$ is skew-symmetric and (7.10) give the components of $\delta\mathbf{C}$. Thus,

$$\delta\mathbf{C}=0 \Leftrightarrow \sum_a g^{aa}(\lambda_1^a\lambda_2^b-\lambda_2^a\lambda_1^b)(\partial_a^2\Gamma_b-\Gamma_a\partial_a\Gamma_b)=0.$$

Remark 7.6: For $\mathbf{K}_2=\mathbf{G}$ and $\mathbf{K}_1=\mathbf{K}$, definition (7.5) and equations (7.6) become

$$\mathbf{C}=\mathbf{K}'\mathbf{D}-\mathbf{D}\mathbf{K}', \quad C^{ab}=g^{aa}g^{bb}(\lambda^a-\lambda^b)D_{ab}, \quad C^a_b=g^{aa}(\lambda^a-\lambda^b)D_{ab}, \tag{7.11}$$

the remaining components being identically zero. Equations (7.7) become

$$\nabla_i C^i_\alpha=0, \quad \nabla_i C^i_b=\sum_a g^{aa}(\lambda^a-\lambda^b)(\partial_a D_{ab}-\Gamma_a D_{ab}+\tfrac{1}{2}\partial_a \ln g^{bb}(D_{ba}-D_{ab})).$$

For $D_{ij}=\partial_i\Gamma_j$ the definition (7.11) is equivalent to

$$\mathbf{C}=\mathbf{K}\mathbf{D}-\mathbf{D}\mathbf{K}$$

and

$$\nabla_i C^i_\alpha=0, \quad \nabla_i C^i_b=\sum_a g^{aa}(\lambda^a-\lambda^b)(\partial_a^2\Gamma_b-\Gamma_a\partial_a\Gamma_b). \tag{7.12}$$

$\mathbf{C}$ is skew-symmetric and (7.12) give the components of $\delta\mathbf{C}$. Thus,

$$\delta\mathbf{C}=0 \Leftrightarrow \sum_a g^{aa}(\lambda^a-\lambda^b)(\partial_a^2\Gamma_b-\Gamma_a\partial_a\Gamma_b)=0. \tag{7.13}$$

Remarks and propositions similar to Remarks 4.6, 4.7, 4.13, 4.14 and Propositions 4.8–12 hold in the present case, with obvious modifications.

VIII. COMMUTATION RELATIONS IN STANDARD COORDINATES

Proposition 8.1: If $\mathbf{K}$ *is a symmetric tensor in standard form (6.1), then the corresponding pseudo-Laplacian assumes the form*

$$\begin{gathered}\Delta_{\mathbf{K}}\psi=A^a\partial_a^2\psi+B^a\partial_a\psi+K^{\alpha\beta}\partial_\alpha\partial_\beta\psi,\\ A^a=K^{aa}=\lambda^a g^{aa},\\ B^a=g^{aa}(\partial_a\lambda^a-\lambda^a\Gamma_a).\end{gathered} \tag{8.1}$$

Proof:

$$\begin{aligned}\Delta_{\mathbf{K}}\psi=\partial_i(K^{ij}\partial_j\psi)+\Gamma^i_{ih}K^{hj}\partial_j\psi&=\partial_i K^{ij}\partial_j\psi+K^{ij}\partial_i\partial_j\psi+\Gamma^i_{ia}K^{aj}\partial_j\psi\\ &=\partial_a K^{aa}\partial_a\psi+K^{aa}\partial_a^2\psi+K^{\alpha\beta}\partial_\alpha\partial_\beta\psi+\Gamma^i_{ia}K^{aa}\partial_a\psi.\end{aligned}$$

Then (8.1) follow from (7.8) and $K^{aa}=\lambda^a g^{aa}$. ■

Remark 8.2: For a pseudo-Laplacian we use the decomposition

$$\Delta_{\mathbf{K}}=\Delta'_{\mathbf{K}}+\Delta''_{\mathbf{K}},$$

$$\Delta'_{\mathbf{K}}\psi=A^a\partial_a^2\psi+B^a\partial_a\psi=g^{aa}(\lambda^a\partial_a^2\psi+(\partial_a\lambda^a-\lambda^a\Gamma_a)\partial_a\psi),$$

$$\Delta''_{\mathbf{K}}\psi=K^{\alpha\beta}\partial_\alpha\partial_\beta\psi.$$

Note that, in accordance with the decomposition (7.1), we have

$$\Delta'_{\mathbf{K}}=\Delta_{\mathbf{K}'}.$$

For a Killing tensor,

$$\Delta'_{\mathbf{K}} = \lambda^a g^{aa}(\partial_a^2\psi - \Gamma_a \partial_a \psi).$$

For a second-order operator (2.12) we use the decomposition

$$\hat{H}_{\mathbf{K}} = \hat{H}'_{\mathbf{K}} + \hat{H}''_{\mathbf{K}}, \quad \hat{H}'_{\mathbf{K}} = -\frac{\hbar^2}{2}\Delta'_{\mathbf{K}} + V_{\mathbf{K}}, \quad \hat{H}''_{\mathbf{K}} = -\frac{\hbar^2}{2}\Delta''_{\mathbf{K}},$$

so that

$$\begin{aligned}\hat{H}_{\mathbf{K}}\psi &= -\frac{\hbar^2}{2}(\Delta'_{\mathbf{K}} + \Delta''_{\mathbf{K}})\psi + V_{\mathbf{K}}\psi \\ &= -\frac{\hbar^2}{2}(A^a \partial_a^2 \psi + B^a \partial_a \psi + K^{\alpha\beta}\partial_\alpha \partial_\beta \psi) + V_{\mathbf{K}}\psi \\ &= -\frac{\hbar^2}{2}[g^{aa}(\lambda^a \partial_a^2 \psi + (\partial_a \lambda^a - \lambda^a \Gamma_a)\partial_a \psi) + K^{\alpha\beta}\partial_\alpha\partial_\beta\psi] + V_{\mathbf{K}}\psi. \end{aligned} \tag{8.2}$$

For a Killing tensor,

$$\hat{H}_{\mathbf{K}}\psi = -\frac{\hbar^2}{2}[\lambda^a g^{aa}(\partial_a^2\psi - \Gamma_a\partial_a\psi) + K^{\alpha\beta}\partial_\alpha\partial_\beta\psi] + V_{\mathbf{K}}\psi.$$

In all the above-given expressions the coordinates (q^α) are ignorable.

Proposition 8.3: The commutator of two second-order operators of the kind (8.2) has the following expression:

$$\begin{aligned}[\hat{H}_{\mathbf{K}_1}, \hat{H}_{\mathbf{K}_2}]\psi = [\hat{H}'_{\mathbf{K}_1}, \hat{H}'_{\mathbf{K}_2}]\psi + \frac{\hbar^4}{2}(A_1^a \partial_a^2 K_2^{\alpha\beta} - A_2^a \partial_a^2 K_1^{\alpha\beta} + B_1^a \partial_a K_2^{\alpha\beta} - B_2^a \partial_a K_1^{\alpha\beta})\partial_{\alpha\beta}\psi \\ + \hbar^4 (A_1^a \partial_a K_2^{\alpha\beta} - A_2^a \partial_a K_1^{\alpha\beta})\partial_a \partial_\alpha \partial_\beta \psi,\end{aligned} \tag{8.3}$$

where

$$[\hat{H}'_{\mathbf{K}_1}, \hat{H}'_{\mathbf{K}_2}]\psi = \frac{\hbar^4}{4}[\Delta'_{\mathbf{K}_1}, \Delta'_{\mathbf{K}_2}]\psi - \hbar^2 (A_1^a \partial_a V_{\mathbf{K}_2} - A_2^a \partial_a V_{\mathbf{K}_1})\partial_a \psi - \frac{\hbar^2}{2}(\Delta'_{\mathbf{K}_1} V_{\mathbf{K}_2} - \Delta'_{\mathbf{K}_2} V_{\mathbf{K}_1})\psi. \tag{8.4}$$

Proof: Since

$$\begin{aligned}\Delta'_{\mathbf{K}_1}\Delta'_{\mathbf{K}_2}\psi &= A_1^b \partial_b^2 (A_2^a \partial_a^2 \psi + B_2^a \partial_a \psi) + B_1^b \partial_b (A_2^a \partial_a^2 \psi + B_2^a \partial_a \psi) \\ &= A_1^b A_2^a \partial_b^2 \partial_a^2 \psi + A_1^b \partial_b^2 A_2^a \partial_a^2 \psi + 2A_1^b \partial_b A_2^a \partial_b \partial_a^2 \psi + A_1^b \partial_b^2 B_2^a \partial_a \psi + A_1^b B_2^a \partial_b^2 \partial_a \psi \\ &\quad + 2A_1^b \partial_b B_2^a \partial_a \partial_b \psi + B_1^b A_2^a \partial_b \partial_a^2 \psi + B_1^b \partial_b A_2^a \partial_a^2 \psi + B_1^b B_2^a \partial_b \partial_a \psi + B_1^b \partial_b B_2^a \partial_a \psi,\end{aligned}$$

we have

$$\begin{aligned}[\Delta'_{\mathbf{K}_1}, \Delta'_{\mathbf{K}_2}] &= 2(A_1^a \partial_a A_2^b - A_2^a \partial_a A_1^b)\partial_a \partial_b^2 \psi + (A_1^a \partial_a^2 A_2^b - A_2^a \partial_a^2 A_1^b + B_1^a \partial_a A_2^b - B_2^a \partial_a A_1^b)\partial_b^2 \psi \\ &\quad + 2(A_1^a \partial_a B_2^b - A_2^a \partial_a B_1^b)\partial_a \partial_b \psi + (A_1^a \partial_a^2 B_2^b + B_1^a \partial_a B_2^b - A_2^a \partial_a^2 B_1^b - B_2^a \partial_a B_1^b)\partial_b \psi.\end{aligned}$$

Since

$$\Delta''_{\mathbf{K}_1}, \Delta''_{\mathbf{K}_2}\psi = K_1^{\alpha\beta} K_2^{\mu\nu} \partial_{\alpha\beta\mu\nu}\psi,$$

we have

$$[\Delta''_{\mathbf{K}_1},\Delta''_{\mathbf{K}_2}]=0,\quad [\hat{H}''_{\mathbf{K}_1},\hat{H}''_{\mathbf{K}_2}]=0.$$

Thus,

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=[\hat{H}'_{\mathbf{K}_1},\hat{H}'_{\mathbf{K}_2}]+[\hat{H}'_{\mathbf{K}_1},\hat{H}''_{\mathbf{K}_2}]+[\hat{H}''_{\mathbf{K}_1},\hat{H}'_{\mathbf{K}_2}].$$

A straightforward calculation shows that

$$[\hat{H}'_{\mathbf{K}_1},\hat{H}'_{\mathbf{K}_2}]\psi=\frac{\hbar^4}{2}(A_1^a\partial_a^2K_2^{\alpha\beta}+B_1^a\partial_aK_2^{\alpha\beta})\partial_\alpha\partial_\beta\psi+\hbar^4A_1^a\partial_aK_2^{\alpha\beta}\partial_a\partial_\alpha\partial_\beta\psi.$$

These two last equations prove (8.3) and (8.4). ■

Proposition 8.4: Let $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be symmetric tensors in standard form. Then* $[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0$ *if and only if*

$$A_1^a\partial_aA_2^b-A_2^a\partial_aA_1^b=0\quad (a,\ \text{n.s.}),\tag{8.5a}$$

$$\sum_a (A_1^a\partial_a^2A_2^b-A_2^a\partial_a^2A_1^b+B_1^a\partial_aA_2^b-B_2^a\partial_aA_1^b)+2(A_1^b\partial_bB_2^b-A_2^b\partial_bB_1^b)=0,\tag{8.5b}$$

$$A_1^a\partial_aB_2^b-A_2^a\partial_aB_1^b+A_1^b\partial_bB_2^a-A_2^b\partial_bB_1^a=0\quad (a\neq b\ \ \text{n.s.}),\tag{8.5c}$$

$$\frac{\hbar^2}{4}\sum_a (A_1^a\partial_a^2B_2^b+B_1^a\partial_aB_2^b-A_2^a\partial_a^2B_1^b-B_2^a\partial_aB_1^b)-A_1^b\partial_bV_{\mathbf{K}_2}+A_2^b\partial_bV_{\mathbf{K}_1}=0\quad (b\ \ \text{n.s.}),\tag{8.5d}$$

$$\Delta'_{\mathbf{K}_1}V_{\mathbf{K}_2}-\Delta'_{\mathbf{K}_2}V_{\mathbf{K}_1}=0,\tag{8.5e}$$

and

$$A_1^a\partial_a^2K_2^{\alpha\beta}-A_2^a\partial_a^2K_1^{\alpha\beta}+B_1^a\partial_aK_2^{\alpha\beta}-B_2^a\partial_aK_1^{\alpha\beta}=0,\tag{8.6a}$$

$$A_1^a\partial_aK_2^{\alpha\beta}-A_2^a\partial_aK_1^{\alpha\beta}=0.\tag{8.6b}$$

Proof: Equations (8.6) follow from the second and third term in (8.3) i.e., from the vanishing of the coefficients of $\partial_\alpha\partial_\beta\psi$ and of $\partial_a\partial_\alpha\partial_\beta\psi$. The first term (8.3) involves only the partial derivatives ∂_a and a factor of ψ, and it is similar (with an obvious change of indices) to (5.3). Thus, (8.5) are similar to (5.5).

Remark 8.5: Since,

$$\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=2(A_1^a\partial_1^aA_2^b-A_2^a\partial_aA_1^b)p_ap_b^2+2(A_1^a\partial_aK_2^{\alpha\beta}-A_2^a\partial_aK_1^{\alpha\beta})p_ap_\alpha p_\beta,\tag{8.7}$$

Eqs. (8.5a) and (8.6a) are equivalent to $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$. Thus,

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0\ \Rightarrow\ \{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0.\tag{8.8}$$

Theorem 8.6: *Let* $\mathbf{K}$ *be a symmetric tensor in standard form. Then the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}},\hat{H}]=0 \Leftrightarrow \begin{array}{c}\{P_{\mathbf{K}},P_{\mathbf{G}}\}=0 \quad (\mathbf{K}\ \text{Killing tensor}),\\ \frac{\hbar^2}{4}\,\delta\mathbf{C}+\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}=0\end{array} \Leftrightarrow \{H_{\mathbf{K}},H\}=-\frac{\hbar^2}{4}P_{\delta\mathbf{C}}, \tag{8.9}$$

where

$$\mathbf{C}=\mathbf{KD}-\mathbf{DK}=\mathbf{K}'\mathbf{D}-\mathbf{DK}', \quad \mathbf{D}=(\partial_i\Gamma_j). \tag{8.10}$$

Proof: The equivalence of the two definitions (8.10) of $\mathbf{C}$ follows from Remarks 7.5 and 7.6. We use (8.5), Proposition 8.4, for $\mathbf{K}_1=\mathbf{K}$ and $\mathbf{K}_2=\mathbf{G}$. Assume $[\hat{H}_{\mathbf{K}},\hat{H}]=0$. From (8.8) it follows that $\mathbf{K}$ is a Killing tensor. For a Killing tensor in standard form we have formulas similar to (5.11) and (5.12), with indices (a,b). Equations (8.5a, b) and (8.6b) are then identically satisfied. The remaining equations are similar to (5.13),

$$(\lambda^a-\lambda^b)(\partial_a\Gamma_b-\partial_b\Gamma_a)=0,$$

$$\frac{\hbar^2}{4}\sum_a g^{aa}(\lambda^a-\lambda^b)(\partial_a^2\Gamma_b-\Gamma_a\partial_a\Gamma_b)+\lambda^b\partial_b V-\partial_b V_{\mathbf{K}}=0 \quad (b\ \text{ n.s.}), \tag{8.11}$$

$$\delta(\mathbf{K}\nabla V-\nabla V_{\mathbf{K}})=0.$$

Due to (7.3), the first equation is identically satisfied. According to Remark 7.6 and Eq. (7.12), the second equation (8.11) is equivalent to

$$\frac{\hbar^2}{4}\,\delta\mathbf{C}+\mathbf{K}\nabla V-\nabla V_{\mathbf{K}}=0,$$

where $\mathbf{C}$ is skew-symmetric. Since $\delta^2\mathbf{C}=0$, the last equation (8.11) is a consequence of the second equation (8.11). The above-noted reasoning is reversible, and the first equivalence (8.9) is proved. The second equivalence follows from the last equation (2.3). ∎

Theorem 8.7: *Let* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be Killing tensors in standard form. Then* $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$, *and the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}_1},\hat{H}_{\mathbf{K}_2}]=0 \Leftrightarrow \frac{\hbar^2}{4}\,\delta\mathbf{C}+\mathbf{K}_1\nabla V_{\mathbf{K}_2}-\mathbf{K}_2\nabla V_{\mathbf{K}_1}=0 \Leftrightarrow \{H_{\mathbf{K}_1},H_{\mathbf{K}_2}\}=-\frac{\hbar^2}{4}P_{\delta\mathbf{C}}, \tag{8.12}$$

where

$$\mathbf{C}=\mathbf{K}_1\mathbf{DK}_2-\mathbf{K}_2\mathbf{DK}_1=\mathbf{K}_1'\mathbf{DK}_2'-\mathbf{K}_2'\mathbf{DK}_1', \quad \mathbf{D}=(\partial_i\Gamma_j). \tag{8.13}$$

Proof: The equivalence of the two definitions of $\mathbf{C}$ in (8.13) follows from Remark 7.5. The components of $\mathbf{C}$ are given in (7.9). The involutivity condition $\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}=0$ follow from (8.7), (8.5a), (8.6b), and (7.2). We use Eqs. (8.5) and (8.6). For Killing tensors $\mathbf{K}_I$ $(I=1,2)$ in standard form we have, cf. (8.1) and (7.2),

$$A_1^a=\lambda_I^a g^{aa}, \quad B_I^a=-g^{aa}\lambda_I^a\Gamma_a,$$

$$\partial_a K_I^{\alpha\beta}=\lambda_I^a\partial_a g^{\alpha\beta}, \quad \partial_a^2 K^{\alpha\beta}=\lambda_I^a\partial_a^2 g^{\alpha\beta}.$$

Thus, (8.6) are identically satisfied. Moreover, formulas similar to (5.17) hold with indices (a,b),

$$A_I^a = \lambda_I^a g^{aa}, \quad B_I^a = -A_I^a \Gamma_a,$$

$$\partial_a A_I^b = A_I^a \partial_a g^{bb} = \lambda_I^a \partial_a g^{bb},$$

$$\partial_a^2 A_I^b = A_I^a \partial_a^2 g^{bb} = \lambda_I^a \partial_a^2 g^{bb},$$

$$\partial_a B_I^b = -\partial_a (A_I^b \Gamma_b) = -\partial_a A_I^b \Gamma_b - A_I^b \partial_a \Gamma_b,$$

$$\partial_a^2 B_I^b = -\partial_a^2 A_I^b \Gamma_b - 2\partial_a A_I^b \partial_a \Gamma_b - A_I^b \partial_a^2 \Gamma_b.$$

Since the coordinates q^α are ignorable and no greek index is involved in (8.5) the proof of the first equivalence (8.12) is similar to that of the first equivalence (5.15) in Theorem 5.8. The second equivalence (8.12) follows from the first equation (2.3) (cf. the end of the proof of Theorem 5.8). ■

Proposition 8.8: Let $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be Killing tensors in standard form. Then*

$$[\hat{H}_{\mathbf{K}_1}, \hat{H}] = 0, \quad [\hat{H}_{\mathbf{K}_2}, \hat{H}] = 0 \Rightarrow [\hat{H}_{\mathbf{K}_1}, \hat{H}_{\mathbf{K}_2}] = 0.$$

Proof: This implication is similar to (5.19) of Proposition 5.9. The proof follows the same pattern. ■

Propositions similar to Propositions 5.10 and 5.11 hold.

Proposition 8.9: Let $\mathcal{H} = (\mathcal{K}, V)$ *be the space of quadratic first integrals in involution associated with the separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}}, \hat{H}] = 0, \quad \forall \mathbf{K} \in \mathcal{K},$$

$$\delta(\mathbf{K}'\mathbf{R} - \mathbf{R}\mathbf{K}') = 0, \quad \forall \mathbf{K} \in \mathcal{K}, \tag{8.14}$$

$$\partial_a R_{ab} - \Gamma_a R_{ab} = 0, \quad (a \neq b \ \text{ n.s.}).$$

Proof: We apply the equivalence of the first and last conditions (8.9) in Theorem 8.6. Since $H_{\mathbf{K}}$ are first integrals, the commutation relation $[\hat{H}_{\mathbf{K}}, \hat{H}] = 0$ is equivalent to $\delta\mathbf{C} = 0$ with $\mathbf{C} = \mathbf{KD} - \mathbf{DK} = \mathbf{K}'\mathbf{D} - \mathbf{DK}'$ and $\mathbf{D} = (\partial_i \Gamma_j)$. If we consider $\mathbf{C} = \mathbf{K}'\mathbf{D} - \mathbf{DK}'$ then only the components $D_{ab} = \partial_a \Gamma_b$ with $a \neq b$ are involved and, since the coordinates are separable, we can replace $\mathbf{D}$ by $\frac{2}{3}\mathbf{R}$, because of (8.13) and (6.4). This proves the equivalence of the first two conditions (8.14). On the other hand, due to Remark 7.5 and (6.4), in the equivalence (7.13) we can replace $\partial_a \Gamma_b$ by R_{ab}, since only the indices $a \neq b$ are involved. This proves the equivalence between $\delta\mathbf{C} = 0$ and the last condition (8.14). ■

Proposition 8.10: Let $\mathcal{H} = (\mathcal{K}, V)$ *be the space of quadratic first integrals in involution associated with the separation of the Hamilton–Jacobi equation. Then the following conditions are equivalent:*

$$[\hat{H}_{\mathbf{K}_1}, \hat{H}_{\mathbf{K}_2}] = 0, \quad \forall \mathbf{K}_1, \mathbf{K}_2, \in \mathcal{K},$$

$$\delta(\mathbf{K}_1'\mathbf{R}\mathbf{K}_2' - \mathbf{K}_2'\mathbf{R}\mathbf{K}_1') = 0, \quad \forall \mathbf{K}_1, \mathbf{K}_2, \in \mathcal{K}, \tag{8.15}$$

$$\partial_a R_{ab} - \Gamma_a R_{ab} = 0, \quad a \neq b.$$

The proof is similar to that of Proposition 5.11. The last condition (8.15) also appears in (8.14). Thus all these conditions are equivalent. For proving Theorem 6.1 it remains to prove that

$$\delta(\mathbf{K}'\mathbf{R} - \mathbf{R}\mathbf{K}') = \delta(\mathbf{KR} - \mathbf{RK}) \tag{8.16}$$

and

$$\delta(\mathbf{K}_1'\mathbf{R}\mathbf{K}_2'-\mathbf{K}_2'\mathbf{R}\mathbf{K}_1')=\delta(\mathbf{K}_1\mathbf{R}\mathbf{K}_2-\mathbf{K}_2\mathbf{R}\mathbf{K}_1). \tag{8.17}$$

These equalities can be proved by the following general considerations on the tensors in *prestandard form*. We say that a contravariant two-tensor $\mathbf{T}=(T^{ij})$ has a prestandard form with respect to a standard coordinate system $(q^i)=(q^a,q^\alpha)$ if $T^{a\alpha}=T^{\alpha a}=0$ and all the remaining components do not depend on the ignorable coordinates (q^α).

Proposition 8.11: For a tensor in prestandard form $\nabla_i T^{i\alpha}=0$.

Proof: Since (q^α) are ignorable, $\partial_i T^{i\alpha}=\partial_a T^{a\alpha}=0$ and we have

$$\nabla_i T^{i\alpha}=\partial_i T^{i\alpha}+\Gamma^i_{il}T^{l\alpha}+\Gamma^\alpha_{il}T^{il}=\Gamma^i_{i\beta}T^{\beta\alpha}+\Gamma^\alpha_{il}T^{il}.$$

However, in standard coordinates $\Gamma^i_{i\beta}=0$ and Γ^α_{il} are all vanishing except for $(i,l)=(a,\beta)$ or (β,a). ■

Proposition 8.12: If $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are in prestandard form, then also the commutator* $\mathbf{C}=\mathbf{T}_1\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1$ *is in prestandard form.*

Proof: By definition of commutator,

$$C^{ij}=T_1^{il}g_{lm}T_2^{mj}-T_2^{il}g_{lm}T_1^{mj}=T_1^{ic}g_{cd}T_2^{dj}+T_1^{i\mu}g_{\mu\nu}T_2^{\nu j}-T_2^{ic}g_{cd}T_1^{dj}-T_2^{i\mu}g_{\mu\nu}T_1^{\nu j},$$

so that

$$C^{ab}=T_1^{ac}g_{cd}T_2^{db}-T_2^{ac}g_{cd}T_1^{db}=T_1^{ac}g_{cc}T_2^{cb}-T_2^{ac}g_{cc}T_1^{cb},$$

$$C^{a\alpha}=C^{\alpha a}=0, \tag{8.18}$$

$$C^{\alpha\beta}=T_1^{\alpha\mu}g_{\mu\nu}T_2^{\nu\beta}-T_2^{\alpha\mu}g_{\mu\nu}T_1^{\nu\beta}.$$

■

For a tensor in a prestandard form let us use the decomposition

$$\mathbf{T}=\mathbf{T}'+\mathbf{T}''=T^{ab}\partial_a\otimes\partial_b+T^{\alpha\beta}\partial_\alpha\otimes\partial_\beta.$$

By setting $T_1^{\alpha\beta}=0$ or $T_2^{\alpha\beta}=0$ in (8.18) we get

$$\mathbf{T}_1'\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1'=\mathbf{T}_1\mathbf{T}_2'-\mathbf{T}_2'\mathbf{T}_1=\mathbf{T}_1'\mathbf{T}_2'-\mathbf{T}_2'\mathbf{T}_1'=\mathbf{C}'.$$

Proposition 8.13: If $\mathbf{C}$ *is a skew-symmetric tensor in prestandard form then* $\delta\mathbf{C}=\delta\mathbf{C}'$.

Proof: Since also $\mathbf{C}'$ is in prestandard form, due to Proposition 8.11 we have $\nabla_i C^{i\alpha}=\nabla_i C'^{i\alpha}=0$. Moreover, $\nabla_i C^{ia}=\partial_i C^{ia}+\Gamma^i_{il}C^{la}+\Gamma^a_{il}C^{il}=\partial_b C^{ba}+\Gamma^i_{ib}C^{ba}$, since $\Gamma^i_{i\alpha}=0$, $\Gamma^a_{il}=\Gamma^a_{li}$ and $C^{il}=-C^{li}$. In this last expression the components $C^{\alpha\beta}$ are not involved. Thus, $\nabla_i C^{ia}=\nabla_i C'^{ia}$. ■

Proposition 8.14: If $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are symmetric tensors in prestandard form, then*

$$\delta(\mathbf{T}_1'\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1')=\delta(\mathbf{T}_1\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1).$$

Proof: The commutator $\mathbf{C}=\mathbf{T}_1\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1$ is skew-symmetric and in prestandard form (Proposition 8.12). The same for $\mathbf{T}_1'\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_1'=\mathbf{C}'$. Then we apply Proposition 8.13. ■

For $\mathbf{T}_1=\mathbf{K}$ (which is in standard form) and $\mathbf{T}_2=\mathbf{R}$ (which is in prestandard form) we get (8.16).

Proposition 8.15: If $\mathbf{T}$, $\mathbf{T}_1$, *and* $\mathbf{T}_2$ *are symmetric tensors in prestandard form, then*

$$\delta(\mathbf{T}_1'\mathbf{T}\mathbf{T}_2'-\mathbf{T}_2'\mathbf{T}\mathbf{T}_1')=\delta(\mathbf{T}_1\mathbf{T}\mathbf{T}_2-\mathbf{T}_2\mathbf{T}\mathbf{T}_1).$$

Proof: The components of $\mathbf{C}=\mathbf{T}_1\mathbf{T}\mathbf{T}_2-\mathbf{T}_2\mathbf{T}\mathbf{T}_1$ are

$$C^{ij}=T_1^{il}T_{lm}T_2^{mj}-T_2^{il}T_{lm}T_1^{mj}=T_1^{ic}T_{cd}T_2^{dj}+T_1^{i\mu}T_{\mu\nu}T_2^{\nu j}-T_2^{ic}T_{cd}T_1^{dj}-T_2^{i\mu}T_{\mu\nu}T_1^{\nu j},$$

so that

$$C^{ab}=T_1^{ac}T_{cd}T_2^{db}-T_2^{ac}T_{cd}T_1^{db},$$

$$C^{a\alpha}=C^{\alpha a}=0,$$

$$C^{\alpha\beta}=T_1^{\alpha\mu}T_{\mu\nu}T_2^{\nu\beta}-T_2^{\alpha\mu}T_{\mu\nu}T_1^{\nu\beta}.$$

This shows that **C** is skew-symmetric and in prestandard form. By setting $T_1^{\alpha\beta}=T_2^{\alpha\beta}=0$ we get $C^{\alpha\beta}=0$. This shows that $\mathbf{T}_1'\mathbf{T}\mathbf{T}_2'-\mathbf{T}_2'\mathbf{T}\mathbf{T}_1'=\mathbf{C}'$. Then we apply Proposition 8.13. ■

For $\mathbf{T}_1=\mathbf{K}_1$, $\mathbf{T}_2=\mathbf{K}_2$, and $\mathbf{T}=\mathbf{R}$ we get (8.17). This completes the proof of Theorem 6.1.

IX. FINAL REMARKS

In this paper we have considered the symmetry operators corresponding to the separation of the Schrödinger equation, but deeper and wider research on this topic still has to be done. Indeed, we have not included here a revisitation of the *R*-separation, leading to a different development of the separation of variables for both Schrödinger and Hamilton–Jacobi equations. This will be the subject of a future paper. A further topic to be investigated is the link between the commutation relations of second-order polynomial observables $H_{\mathbf{K}}$ and the associated second-order operators $\hat{H}_{\mathbf{K}}$, for generic two-tensors **K** on Riemannian manifolds. This matter is concerned mainly with integrability of systems with quadratic first integrals, and the separability appears only as a special case.

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